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STABILITY AND STABILITY SWITCHING IN 3-SPECIES MODEL WITH DELAYS

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ABSTRACT

Sufficient conditions for the stability of the positive equilibrium of a 3-species Lotka-Volterra-type system with density-dependent death rates modeling the interactions of two competing predators living exclusively on a common prey with delays, are obtained by constructing a suitable lyapunov function. An estimation of delay length for which equilibrium stability holds is obtained. We show the existence of some critical value $\hat{\sigma}$ of the delay σ in the predation and biomass conversion terms at which the stability switches to instability and the system remains unstable for $\sigma > \hat{\sigma}$.

Key words: Predator-Prey System, Equilibrium, Lyapunov Function, Delay, Stability-Switching.

INTRODUCTION

One of the most interesting questions in ecological models concerns with local stability of the positive equilibrium of the model. The question becomes even more interesting when we consider time delays in the growth dynamics of the interacting populations of various species. Delays can arise from a great variety of causes and undoubtedly always present to some extent. In order to, realistically, model a real life system of inter-specific (and intra-specific) interations, one generally needs to take into account the influence due to the past states of such a system, or the waiting period resulting because of physiological processes. These usually mean that a real system should be modeled by differential with time delays. Indeed, such a view is gaining popularity across the board of science and engineering, as evidenced by the rapid increase in the number of mathematical models of delay differential equations studied in various scientific disciplines, e.g., see the numerous references cited in [1]. This is specially true in the area of mathematical biology, see [4,8,11].

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Considering the historical Lotka-Volterra 2-dim system without delays, that is

$$\begin{aligned}\frac{dH}{dt} &= aH - bHP \\ \frac{dP}{dt} &= -dH + cHP\end{aligned} \tag{1.1}$$

where a, b, c, d are positive constants, it is very well known that (1.1) is unrealistic for various reasons. We, for simplicity, nondimensionalize (1.1) by

putting $u(\tau) = \frac{cH(t)}{d}, v(\tau) = \frac{bP(t)}{a}, \tau = at, a = \frac{d}{a}$, then (1.1) becomes:

$$\begin{aligned}\frac{du}{d\tau} &= u(1 - v) \\ \frac{dv}{d\tau} &= \alpha v(u - 1)\end{aligned} \tag{1.2}$$

The trajectories of (1.2) are given by

$$\alpha u + v - v ln\left(u^{\alpha}\right) = N, \text{where} N > 1 + \alpha, \tag{1.3}$$

and in the phase plane are closed (periodic solutions) as shown in Fig. (1.1), a typical periodic solution $(u(\tau), v(\tau))$ is shown in Fig. (1.2). A major inadequacy of (1.1) is that its solutions are not structurally stable. Suppose for example $u(0)$ and

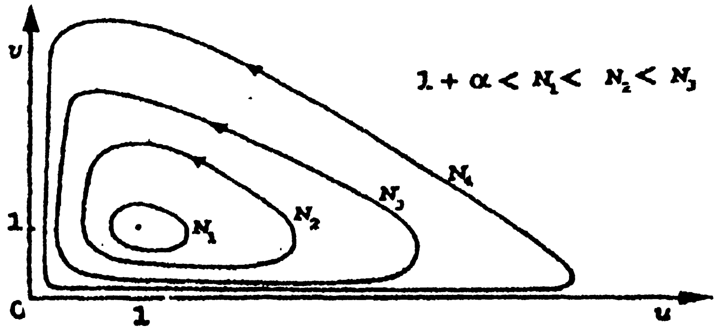


***Fig.* (1.2)**

$v(0)$ are such that u and v for $\tau > 0$ are on the trajectory N_4 which passes close to the u and v axes. Then any small perturbation will move the solution onto another trajectory which does not lie everywhere close to the original one N_4. Thus a small perturbation can have a very marked effect, at the very least on the amplitude of the oscillation. In other words, this high sensitivity to the smallest change in the model leads to a model with essentially different species behaviour. For instance the positive equilibrium point of the system which is a center changes, under small pertubations, to a focus. For a 3-species Lotka-Volterra-type model without delays, as show in [5], there exist Hopf-bifurcating periodic orbits, but no conclusion about their orbital stability can be drawn by using the available procedures given in [10, 13] and others.

Thus, instead of relying on the instantanous density of one species (prey, say), because young preys do not participate in the reproductive process prior to sexual maturity, together with the fact that the predators are not usually successful in hunting young preys which do not wander away from their safe havens, we can get a more realistic model if we take into account density of one species (prey say) in the past, that is a system with delays.

There are several examples of 3-species communities such as linear chain of 3-species, see Fig. (1.3) (i); two preys and one predator, see Fig. (1.3) (ii) and two predators interacting with one prey as in Fig. (1.3) (iii).

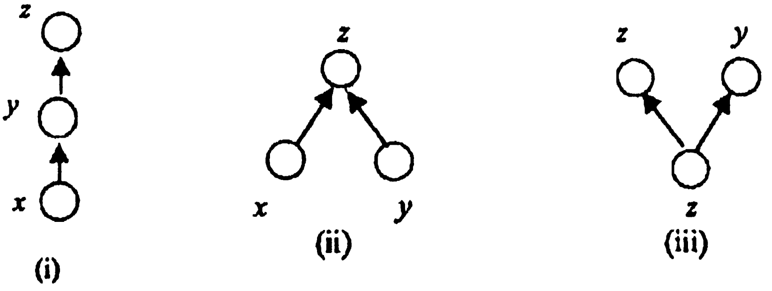


Fig. (1.3)

Since the question of two predatory populations consuming a single prey population has occupied an important place in ecological literature, thus in this paper we have a differential system modeling the interaction of two predators feeding upon a common prey. We have incorporated discrete delays $\tau \geq 0$ in the logistic maturation term of the prey species and $\sigma > 0$ in the predation terms and biomass conversion terms. Both predatury populations have been considered with density-dependent death rates. By using Lyapunov function technique as discussed for 2-dim predator-prey system in [9], we have obtained an estimation of delay's length for which stability is preserved, we have shown the existence of a critical value $\hat{\sigma}$ of σ (with $\tau = 0$) at which stability of the unique positive equilibrium switches to instability and the system remains unstable for $\sigma > \hat{\sigma}$.

MODEL WITHOUT DELAYS AND LOCAL ASYMPTOTIC STABILITY

An intermediate type 3-species system, see [7], modeling the inter-species interaction of two predators and one prey is of the form:

$$\left.\begin{aligned} \dot{x} &= xf(x) - yq_1(x) - zq_2(x) \\ \dot{y} &= y[-g(y) + cq_1(x)] \\ \dot{z} &= z[-h(z) + dq_2(x)] \end{aligned}\right\} \tag{2.1}$$

with

$$x(0) = x_0 > 0 \quad , \quad y(0) = y_0 > 0 \quad , \quad z(0) = z_0 > 0 \quad , \quad \cdot = \frac{d}{dt}$$

$f(x)$ is the specific growth rate of the prey x and generally is assumed to have the following properties: $f[0,\infty) \rightarrow R; f \in C^1[0,\infty), f'(x) < 0$; there exists $k>0$ such that $f(k)=0$ where k is the environmental carrying capacity with respect to the prey x. $q_1(x)$ and $q_2(x)$ are the functional responses of predators y and z repectively with respect to the prey x.

Generally, it is assumed that $q_1 ; [0,\infty) \rightarrow \rightarrow [0,\infty); q_i \in C^1[0,\infty); q_i(0) = 0, q_i'(x) > 0$ $i=1,2$.

$g(y)$ and $h(z)$ are the density-dependent specific death rates of predators y and z respectively. It is assumed that $g(0)>0$, $h(0)>0$, $g'(y) \geq 0$ for $y > 0$ and $h'(z) \geq 0$ for $z > 0$.

In this paper, however, we consider the folliwng special case of (2.1):

$$\left.\begin{aligned}\dot{x} &= x\left[\in\left(1-\frac{x}{k}\right)-ay-bz\right]\\ \dot{y} &= y\left[-(1-y)+\frac{1}{a}x\right]\\ \dot{z} &= z\left[-(1+z)+\frac{1}{b}x\right]\end{aligned}\right\} \qquad (2.2)$$

where the parameter $k>0$ is the above mentioned carrying capacity of the environment with respect to the prey x which, in turn, has a logistic growth rate, (i.e. it does not grow unboundedly in the absence of predators), $\in > 0$ is the intrinsic growth rate of the prey x in the absence of predation, $a>0$, $b>0$ are predation rates (i.e. functional responses of predators y and z respectively with respect to x), $\frac{1}{a}, \frac{1}{b}$ are biomass conversion rates. We, further; make the following assumptions:

$$\left.\begin{array}{llll} & 0<a<b<1 & , & k, \in > b > a, \\ \text{and} & ka - \in b > 0 & if & \in \neq k \\ \text{and} & k + a > 2b & if & \in = k. \end{array}\right\} \qquad (2.3)$$

The unique positive equilibrium $E\left(x^*, y^*, z^*\right)$ of (2.2) is given by:

$$\left.\begin{aligned} & x^* = \frac{k(\in + a + b)}{\in + 2k}, \quad y^* = \frac{\in(k-a)+k(b-a)}{a(\in + 2k)}, \quad and \\ & z^* = \frac{k(\in - b) + ka - b\in}{b(\in + 2k)} \end{aligned}\right\} \qquad (2.4)$$

It is straightforward to show that the eigenvalues of the variational matrix $V(E)$ of the system (2.2) about $E(x^*, y^*, z^*)$ are given by the characteristic equation:

$$\lambda^3 + p_1\lambda^2 + p_2\lambda + P_3 = 0 \qquad (2.5)$$

where

$$\left.\begin{aligned}
p_1 &= x^* \frac{\in}{k} + y^* + z^* \\
p_2 &= x^* y^* \left(1 + \frac{\in}{k}\right) + x^* z^* \left(1 + \frac{\in}{k}\right) + y^* z^* \\
p_3 &= x^* y^* z^* \left(2 + \frac{\in}{k}\right).
\end{aligned}\right\} \tag{2.6}$$

Since $P_1 > 0$, $P_3 > 0$, and

$$\begin{aligned}
p_1 p_2 - p_3 = {} & \frac{\in}{k}\left(1 + \frac{\in}{k}\right)\left(x^*\right)^2 \left(y^* + z^*\right) + x^*\left(1 + \frac{\in}{k}\right)\left[\left(y^*\right)^2 + \left(z^*\right)^2\right] \\
& + y^* z^* \left(y^* + z^*\right) + \frac{2 \in}{k} x^* y^* z^* > 0,
\end{aligned}$$

then by Routh-Hurwitz criterion $E\left(x^*, y^*, z^*\right)$ given by (2.4) is locally asymptotically stable.

MODEL WITH DELAYS AND LOCAL ASYMPTOTIC STABILITY

We incorporate discrete delays in the system (2.2) as follows:

$$\left.\begin{aligned}
\dot{x} &= x(t)\left[\in\left(1 - \frac{x(t-\tau)}{k}\right)\right] - ay(t-\sigma) - bz(t-\sigma) \\
\dot{y} &= y(t)\left[-(1 + y(t)) + \frac{1}{a} x(t-\sigma)\right] \\
\dot{z} &= z(t)\left[-(1 - z(t)) + \frac{1}{b} x(t-\sigma)\right]
\end{aligned}\right\} \tag{3.1}$$

with the initial conditions:

$$\left.\begin{aligned} x(t) &= \varnothing_1(t), t \in [-r,0], \varnothing_1 \in C([-r,0],R^+) \\ y(t) &= \varnothing_2(t), t \in [-r,0], \varnothing_2(t) \in C([-r,0],R^+) \\ z(t) &= \varnothing_3(t), t \in [-r,0], \varnothing_3(t) \in C([-r,0],R^+) \end{aligned}\right| \quad (3.2)$$

where $r = \max\{\tau,\sigma\}$.

Clearly the system (3.1) with (3.2) has a unique solution which exits and is positive for all $t \geq 0$. τ denotes delay incorporated in the logistic growth term of the prey species. To make calculation simpler we have taken the delay in both predation terms in the prey growth rate equation and in the biomass conversion terms in both predators growth rate equations to be the same (σ >0 say), otherwise the analysis technique remains the same with different delays.

Naturally $E(x^* y^* z^*)$ as given by (2.4) remains the unique positive equilibrium of the system (3.1). Let us introduce a new coordinate system through the following substitution:

$$x_1(t) = x(t) - x^*, \qquad y_1(t) = y(t) - y^* \qquad z_1(t) = z(t) - z^*.$$

From (3.1), $(x_1(t), y_1(t), z_1(t))$ satisfies the system:

$$\left.\begin{aligned} \dot{x}_1(t) &= (x_1(t) + x^*)x^*\left[-\frac{\in}{k}x_1(t-\tau) - ay_1(t-\sigma) - bz_1(t-\sigma)\right] \\ \dot{y}_1(t) &= (y_1(t) + y^*)y^*\left[-y_1(t) + \frac{1}{a}x_1(t-\sigma)\right] \\ \dot{z}_1(t) &= (z_1(t) + z^*)z^*\left(-z_1(t) + \frac{1}{b}x_1(t-\sigma)\right) \end{aligned}\right\} \quad (3.3)$$

Remark 3.1.

The stability analysis of $E(x^* y^* z^*)$ is reduced to that of zero solution of (3.3). From (3.3), it follows that the variational system of (3.1) with respect to the positive equilibrium $E(x^{*}\cdot y^{*}z^{*})$ is given by the linearized system of (3.3) as:

$$\left.\begin{aligned}
\dot{x}_1(t) &= x^{*}\left[-\frac{\in}{k}x_1(t-\tau)-ay_1(t-\sigma)-bz_1(t-\sigma)\right]\\
\dot{y}_1(t) &= y^{*}\left[-y_1(t)+\frac{1}{a}x_1(t-\sigma)\right]\\
\dot{z}_1(t) &= z^{*}\left(-z_1(t)+\frac{1}{b}x_1(t-\sigma)\right)
\end{aligned}\right\} \tag{3.4}$$

Now, we can rewrite (3.4) as follows:

$$\left.\begin{aligned}
&\frac{d}{dt}\left[\frac{x_1(t)}{x^{*}}-\frac{\in}{k}\int_{t-\tau}^{t}x_1(s)ds-a\int_{t-\sigma}^{t}y_1ds-b\int_{t-\sigma}^{t}z_1(s)ds\right]=-\frac{\in}{k}x_1(t)-ay_1(t)\underline{b}z_1(t) && (i)\\
&\frac{d}{dt}\left[\frac{y_1(t)}{y^{*}}+\frac{1}{a}\int_{t-\sigma}^{t}x_1(s)ds\right]=y_1(t)+\frac{1}{a}x_1(t) && (ii)\\
&\frac{d}{dt}\left[\frac{z_1(t)}{z^{*}}+\frac{1}{b}\int_{t-\sigma}^{t}x_1(s)ds\right]=\underline{z}_1(t)+\frac{1}{b}x_1(t) && (iii)
\end{aligned}\right\} \tag{3.5}$$

Now, we construct a Lyapunov function $w(t)$ so that the trivial solution (i.e. (0,0,0)) of (3.3) is stable and then we show solution $(x_1(t), y_1(t), z_1(t)) \to (0, 0, 0)$ as $t \to \infty$ which implies that (0,0,0) is a locally asymptotically stable solution of (3.3), which consequently implies that $E(x^{*}\cdot, y^{*}, z^{*})$ is locally asymptotically stable.

Consider $$w(t)=x^{*}w_1(t)+y^{*}a^2w_2(t)+z^{*}b^2w_3(t), \tag{3.6}$$

Where $$w_1(t)=w_{11}(t)+w_{12}(t) \tag{3.7}$$

For $$w_{11}(t)=\left[\frac{x_1(t)}{x^{*}}-\frac{\in}{k}\int_{t-\tau}^{t}x_1(s)ds-a\int_{t-\sigma}^{t}y_1(s)ds-b\int_{t-\sigma}^{t}z_1(s)ds\right]^2 \tag{3.8}$$

$$w_{12}(t)=\left(\frac{\in}{k}+a+b\right)\left[\frac{\in}{k}\int_{t-\tau}^{t}\int_{s}^{t}x_1^2(l)dlds+a\int_{t-\sigma}^{t}\int_{s}^{t}y_1^2(l)dlds+b\int_{t-\sigma}^{t}\int_{s}^{t}z_1^2(l)dlds\right] \quad (3.9)$$

where $w_2(t)=w_{21}(t)+w_{22}(t)$ (3.10)

for $$w_{12}(t)=\left[\frac{y_1(t)}{y^*}+\frac{1}{a}\int_{t-\sigma}^{t}x_1(s)ds\right] \quad (3.11)$$

for $$w_{22}(t)=\left(\frac{1}{2}+\frac{1}{a^2}\right)\int_{t-\sigma}^{t}\int_{s}^{t}x_1^2(l)dlds \quad (3.12)$$

and where $w_3(t)=w_{31}(t)+w_{32}(t)$ (3.13)

for $$w_{31}(t)=\left[\frac{z_1(t)}{z^*}+\frac{1}{b}\int_{t-\sigma}^{t}x_1(s)ds\right]^2 . \quad (3.14)$$

$$w_{32}(t)=\left(\frac{1}{b}+\frac{1}{b^2}\right)\int_{t-\sigma}^{t}\int_{s}^{t}x_1^2(l)dlds \quad (3.15)$$

Remarks 3.2.

Let $X(t)=(x_1(t),y_1(t),z_1(t))$. Thus by $w(t)$ wen mean: $w(t)=w(X)(t)$, and similarly for $w_1(t),w_2(t),w_3(t),w_{11}(t),w_{12}(t),w_{21}(t),w_{22}(t),w_{31}(t)$ and $w_{32}(t)$.

From the defination of $w(t)$, it is clear that $w(t)$ is positive definite.

Next we compute the time derivative of $w(t)$ with respect to (3.4). As shown in the Appendix we get:

$$\frac{dw(t)}{dt}\le x_1^2(t)\frac{\in}{k}\left[-2+x^*\left(\frac{2\in}{k}\tau+(a+b)(\tau+\sigma)\right)+\frac{k}{\in}y^*\sigma(a+2)+\frac{k}{\in}\sigma(b+2)z^*\right]$$
$$+a^2y_1^2(t)\left[-2+\frac{x^*}{a}\left(\frac{\in}{k}(\tau+\sigma)+2\sigma(a+b)\right)+\frac{\sigma}{a}y^*\right]$$
$$+b^2z_1^2(t)\left[-2+\frac{x^*}{b}\left(\frac{\in}{k}(\tau+\sigma)+2\sigma(a+b)\right)+\frac{\sigma}{b}z^*\right]$$

Thus

$$\frac{dw(t)}{dt} \leq -\frac{\in}{k}\alpha_1 x_1^2(t) - a^2\alpha_2 y_1^2(t) - b^2\alpha_3 z_1^2(t), \tag{3.16}$$

where it is assumed that:

$$\left.\begin{aligned}
a_1 &= 2-\left\{x^*\left(\frac{2\in}{k}+a+b\right)\tau+\left[x^*(a+b)+\frac{k}{\in}y^*(a+2)+\frac{k}{\in}z^*(b+2)\right]\sigma\right\}>0 && (i)\\
a_2 &= 2-\left\{\left(\frac{x^*\in}{ak}\right)\tau+\frac{1}{a}\left[x^*\left(\frac{\in}{k}+2a+2b\right)+y^*\right]\sigma\right\}>0 && (ii)\\
a_3 &= 2-\left\{\left(\frac{x^*\in}{bk}\right)\tau+\frac{1}{b}\left[x^*\left(\frac{\in}{k}+2a+2b\right)+z^*\right]\sigma\right\}>0 && (iii)
\end{aligned}\right\} (3.17)$$

let $$a = \min\left\{\frac{\in}{k}a_1, \alpha^2\alpha_2, b^2\alpha_3\right\} \tag{3.18}$$

Thus (3.16) gives:

$$\frac{dw(t)}{dt} \leq -a\left(x_1^2(t)+y_1^2(t)+z_1^2(t)\right). \tag{3.19}$$

From (3.19) it follows that $w(x_1(t), y_1(t), z_1(t))$ as defined by (3.6) is the sought after Lyapunov function. To prove local asymptotic stability of $E(x^*, y^*, z^*)$ it remains to show that:

$$\lim_{t\to\infty}\left[x_1^2(t)+y_1^2(t)+z_1^2(t)\right]=0.$$

Now integrating (3.19) from *T* to *t*, we get

$$w(t)+a\int_T^t\left[x_1^2(s)+y_1^2(s)+z_1^2(s)\right]ds \leq w(T), \qquad t \geq T \tag{3.20}$$

But (3.20) implies that $x_1^2(t)+y^2(t)+z^2(t) \in L[T,\infty)$.

We claim that $\lim_{t\to\infty}\left[x_1^2(t)+y_1^2(t)+z_1^2(t)\right]=0$.

Since $w(t)$ is a decreasing function with strict minimum at (0,0,0), $\lim_{t\to\infty} w(t)$ exists and it finite.

This is because $\lim_{t\to\infty}\left[x_1^2(t)+y_1^2(t)+z_1^2(t)\right]=N\neq 0$, will according to (3.20) lead to a contradiction. Thus it follows that the (0,0,0) solution of (3.3) is locally asymtotically stable provided the following three conditions hold:

$$\left.\begin{array}{ll} x^*\left(\dfrac{2\in}{k}+a+b\right)\tau+\left[x^*(a+b)+y^*\dfrac{k}{\in}(a+2)+z^*\dfrac{k}{\in}(b+2)\right]\sigma<2 & (i) \\ \left(\dfrac{x^*\in}{ak}\right)\tau+\left[\dfrac{x^*}{a}\left(\dfrac{\in}{k}+2a+2b\right)+\dfrac{y^*}{a}\right]\sigma<2 & (ii) \\ \left(\dfrac{x^*\in}{bk}\right)\tau+\left[\dfrac{x^*}{b}\left(\dfrac{\in}{k}+2a+2b\right)+\dfrac{z^*}{b}\right]\sigma<2 & (iii) \end{array}\right\} \quad (3.21)$$

Now, we determine values of τ and σ satisfying the three inequalities (3.2.1) (i), (ii) and (iii).

Since from (2.3) we have $\frac{1}{b}<\frac{1}{a}$, and fromn (2.4) it follows $y^*>z^*$ and thus $\frac{y^*}{a}>\frac{z^*}{b}$, hence it is sufficient to find values τ and σ satisfying (3.2.1) (i) and (ii) only. Thus we have proved the following main theorem:

Theorem (3.1)

Assume that (2.3) holds. If the delays τ and σ satisfy the inequalities (3.2.1) (i) and (3.2.1) (ii), then the positive equilibrium $E\left(x^*,y^*,z^*\right)$ is locally asymtotically stable.

Let us consider the following illustrative example:

Example 3.1.

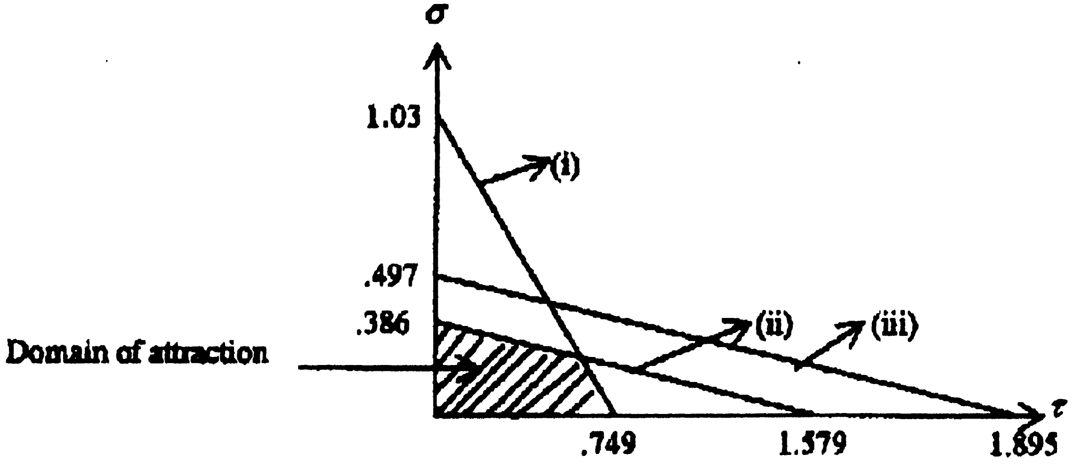
If we let $\in=k=1, a=\frac{5}{8}, b=\frac{6}{8}$, then from (2.4) we have $x^*=\frac{19}{24}, y^*=\frac{4}{15}$, and $z^* = \frac{1}{8}$, consequently the inequalities (3.2.1) (i), (ii) and (iii) become:

$$\left.\begin{aligned} &(2.67)\tau + (1.94)\sigma < 2 && (i)\\ &(1.266)\tau + (5.176)\sigma < 2 && (ii)\\ &(1.055)\tau + (4.022)\sigma < 2 && (iii) \end{aligned}\right\} \tag{3.22}$$

Thus as long as

$$\left.\begin{aligned} &0 \le \tau < .749 \quad \text{and}\\ &0 \le \sigma < .386, \end{aligned}\right\} \tag{3.23}$$

the equilibrium $E\left(\frac{19}{24}, \frac{4}{15}, \frac{1}{18}\right)$ is locally asymtotically stable, see Fig. 3.1.



Remark 3.3.

It follows from (3.23) that the delay $\sigma > 0$ in the predation and biomass conversion terms is more destabilizing than the maturation delay $\tau \ge 0$ of the prey as far as this particular Lotka-Volterra predator-prey model is concerned.

STABILITY SWITCHING

In this section, we show the existence of some critical value $\hat{\sigma} > 0$ of the delay $\sigma > 0$ (with $\tau = 0$) such that for $\sigma > \hat{\sigma}$ there occurs stability switching of the equilibrium $E(x^*, y^*, z^*)$ from stability to instability, in other words the equilibrium $E(x^*, y^*, z^*)$ is unstable for $\sigma > \hat{\sigma}$. The discussion is based upon Theorem 1 of [2] and corollary 4.3. of [6].

We consider the system (3.1) with $\tau = 0$ and $\in = k = 1$, that is

$$\left.\begin{aligned} \dot{x} &= x(t)\left[(1-x(t)) - ay(t-\sigma) - bz(t-\sigma)\right] \\ \dot{y} &= y(t)\left[-(1+y(t)) + \frac{1}{a}x(t-\sigma)\right] \\ \dot{z} &= z(t)\left[-(1+z(t)) + \frac{1}{b}x(t-\sigma)\right] \end{aligned}\right\} \tag{4.1}$$

The characteristic equation for the linearization of (4.1) is

$$\lambda^3 + a\lambda^2 + \beta\lambda + \gamma = (\delta + \lambda 0)e^{-2\lambda\sigma} \tag{4.2}$$

where,

$$\left.\begin{aligned} \alpha &= x^* + y^* + z^* \\ \beta &= x^*(y^* + z^*) + y^*z^* \\ \gamma &= x^*y^*z^* \\ \delta &= -2x^*y^*z^* \\ \theta &= -x^*(y^* + z^*) \end{aligned}\right\} \tag{4.3}$$

Let $P(\lambda) = \lambda^3 + a\lambda^2 + \beta\lambda + \gamma$

$Q(\lambda) = -(\lambda\theta + \delta)$

Since P and Q are polynomials with real coefficients, the condition (HI) of Theorem 1 of [2] is satisfied.

Now

$$P(\lambda) + Q(\lambda) \equiv \lambda^3 + a\lambda^2 + \beta\lambda + \gamma - \lambda\theta - \delta = 0 \quad \text{or}$$

$$\lambda^3 + \left[x^* + y^* + z^*\right]\lambda^2 + \left[2x^*(y^* + z^*) + y^*z^*\right] + 3x^*y^*z^* = 0 \tag{4.4}$$

Now (4.4) is same as (2.5) with (2.6).

Thus as shown in section 2, $E(x^*, y^*, z^*)$ is locally asymptotically stable. This implies that $P(\lambda)+Q(\lambda)=0$ has no root in $\mathrm{Re}(\lambda)\geq 0$. Thus condition (H2) of Theorem 1 of [2] is satisfied. In order to utilize Theorem 1 of [2], let us define:

$$\begin{aligned} F(y) &= |P(iy)|^2 - |Q(iy)|^2 \\ &= y^6 + (\alpha^2 - 2\beta)y^4 + (\beta^2 - 2a\gamma - \theta^2)y^2 + (\gamma^2 - \delta^2) \end{aligned} \tag{4.5}$$

Where

$$\left.\begin{aligned} &a^2 - 2\beta = (x^*)^2 + (y^*)^2 + (z^*)^2 > 0 \\ &F(0) = \gamma^2 - \delta^2 = -3(x^* y^* z^*)^2 < 0 \\ &\beta^2 - 2a\gamma - \theta^2 = y^* z^*\left(y^* z^* - 2(x^*)^2\right) \end{aligned}\right\} \tag{4.6}$$

Now Theorem 1 of [2] is stated as follows:
Assume conditions $(H1)$ and $(H2)$ hold.

(a) If $F(y)$ has no positive roots, then no stability switches occur. (This implies either the equilibrium point is stable for all $\sigma\geq 0$ or it is unstable for all $\sigma\geq 0$.

(b) If $F(y)$ has positive roots, stability switches may occur. At most a finite number of switches is possible. There exists $\hat{\sigma}\geq 0$ or such that the equilibrium point is unstable for all $\sigma\geq\hat{\sigma}$.

Now since $a^2-2\beta>0$ and $\gamma^2-\delta^2<0$, then by Descartes rule of sign applied to $F(y)$ as given by (4.5), the possibility that $F(y)>0$ for all $y>0$ (i.e. $F(y)$ has no positive root) is completely ruled out. This implies that the stability of the equilibrium point $E(x^*,y^*,z^*)$ is not preserved for all $\sigma>0$.

Thus we have following theorem:

Theorem 4.1.

Assume (2.3) (i.e. the equilibrium $E(x^*,y^*,z^*)$ *of (4.1) with* $\sigma=0$ *is asymptotically stable). Then there exists* $\hat{\sigma}>0$ *for which* $E(x^*,y^*,z^*)$ *is unstable for* $\sigma>\hat{\sigma}$.

Remark 4.1.

Prior ultimately shifting to instability for $\sigma>\hat{\sigma}$, there may occur at most a finite number of stability switches.

Example 4.1.

Consider the system (4.1). The associated linearized system has the characteristic equation:

$$\lambda^3 + a\lambda^2 + \beta\lambda + \gamma - (\theta\lambda + \delta)e^{-2\lambda\sigma} = 0.$$

Let $a = \frac{5}{8}, b = \frac{6}{8}, \in = k = 1$, we have $x^* = \frac{19}{24}, y^* = \frac{4}{15}, z^* = \frac{1}{18}$. Thus from (4.3) we get

$a = 1.114,\ \beta = 0.2699,\ \gamma = 0.0117,\ \delta = -0.023457,\ \theta = -0.25499,$ (4.7)

From (4.5), we have:

$$F(y) = y^6 + 0.701119y^4 - 0.018234y^2 - 0.0004107,$$

which has positive real root y_l=0.19724.

This indicates that preservation of stability can not hold for all $\sigma > 0$.

Thus there exists $\hat{\sigma}$ such that for all $\sigma > \hat{0}$, the equilibrium is unstable.

Remarks 4.2.

The delay which could represent in our model influence of the past or waiting period in the proliferation of certain species due to (say) lag in some physiological process (because of natural or unnatural reasons), could have if longer than enough a destabilizing effect on the otherwise stable equilibrium coexistence of the ecosystem's species.

Remark 4.3.

In case the two delays τ and σ are kept in the model (4.1) as in case of (3.1), then the corresponding characteristic equation is:

$$\lambda^3 + (y^* + z^*)\lambda^2 + y^*z^*\lambda + \left[\lambda^2 x^* + \lambda x^*(\lambda^* + z^*) + x^*y^*z^*\right]e^{-\lambda\tau}$$
$$+ \left[\lambda x^*(y^* + z^*) + 2x^*y^*z^*\right]e^{-2\lambda\sigma} = 0.$$

However, discussion of the full analysis of stability switches for arbitary σ and τ remains an open problem, see [2,3].

APPENDIX

In this apprendix we prove (3.16). From (3.8), the time derivative of $w_{l l}(t)$ with respect to (3.4) while using (3.5) is:

$$\frac{dw_{11}(t)}{-dt} = 2\left[\frac{x_1(t)}{x^*} - \frac{\in}{k}\int_{t-\tau}^{t} x_1(s)ds - a\int_{t-\sigma}^{t} y_1(s)ds - b\int_{t-\sigma}^{t} z_1(s)ds\right]$$

$$\times\left[\frac{-\in}{k}x_1(t) - ay_1(t) - bz_1(t)\right]$$

$$= \frac{-2\in}{k}\frac{x_1^2(t)}{x^*} - \frac{2a}{x^*}x_1(t)y_1(t) - \frac{2b}{x^*}x_1(t)z_1(t)$$

$$+2\left(\frac{\in}{k}\right)^2\int_{t-\tau}^{t} x_1(t)x_1(s)ds + \frac{2\in a}{k}\int_{t-\sigma}^{t} x_1(t)y_1(s)ds + \frac{2\in b}{k}\int_{t-\sigma}^{t} x_1(t)z_1(s)ds$$

$$+\frac{2\in a}{k}\int_{t-\tau}^{t} y_1(t)x_1(s)ds + 2a^2\int_{t-\sigma}^{t} y_1(t)y_1(s)ds + 2ab\int_{t-\sigma}^{t} y_1(t)z_1(s)ds$$

$$+\frac{2\in b}{k}\int_{t-\tau}^{t} z_1(t)x_1(s)ds + 2ab\int_{t-\sigma}^{t} z_1(t)y_1(s)ds + 2b^2\int_{t-\sigma}^{t} z_1(t)z_1(s)ds$$

$$\leq \frac{-2\in}{k}\frac{x_1^2(t)}{x^*} - \frac{2a}{x^*}x_1(t)y_1(t) - \frac{2b}{x^*}x_1(t)z_1(t)$$

$$+\left\{\left(\frac{\in^{\delta^2}}{k}\right)\left[x_1^2(t)\tau + \int_{t-\tau}^{t} x_1^2(s)ds\right] + \frac{\in}{k}a\left[x_1^2(t)\sigma + \int_{t-\sigma}^{t} y_1^2(s)ds\right]\right.$$

$$+\frac{\in b}{k}\left[x_1^2(t)\sigma + \int_{t-\sigma}^{t} z_1^2(s)ds\right] + \frac{\in a}{k}\left[y_1^2(t)\tau + \int_{t-\tau}^{t} x_1^2(s)ds\right]$$

$$+a^2\left[y_1^2(t)\sigma + \int_{t-\sigma}^{t} y_1^2(s)ds\right] + ab\left[y_1^2(t)\sigma + \int_{t-\sigma}^{t} z_1^2(s)ds\right]$$

$$\left.+\frac{\in b}{k}\left[z_1^2(t)\tau + \int_{t-r}^{t} x_1^2(s)ds\right] + ab\left[z_1^2(t)\sigma + \int_{t-\sigma}^{t} y_1^2(s)ds\right] + b_2\left[z_1^2(t)\sigma + \int_{t-\sigma}^{t} z_1^2(s)ds\right]\right\} \quad (A-1)$$

Define $w_1(t) = w_1(X)(t) = w_{11}(X)(t) = w_{12}(X)(t)$, (A-2)

Where $w_{12}(t)$ is defined so that the integral terms in (A-1) get eliminated. Thus

$$w_{12}(t) = \left(\frac{\in}{k} + a + b\right)\left[\frac{\in}{k}\int_{t-\tau}^{t}\int_{s}^{t} x_1^2(l)\,dl\,ds + a\int_{t-\sigma}^{t}\int_{s}^{t}\right.$$
$$\left. + b\int_{t-\sigma}^{t}\int_{s}^{t} z_1^2(l)\,dl\,ds\right] \tag{A-3}$$

Using the fact that $\int_{t-r}^{t} x^2(s)ds + \frac{d}{dt}\left[\int_{t-r}^{t}\int_{s}^{t} x^2(l)dlds\right] = x^2(t)r$, then from (A-1), (A-2) and (A-3), we get:

$$\begin{aligned}\frac{dw_1(t)}{dt} \le & \frac{-2\in}{k}\frac{x_1^2(t)}{x^*} - \frac{2a}{x^*}x_1(t)y_1(t) - \frac{2b}{x^*}x_1(t)z_1(t)\\
& + \left(\frac{\in}{k}\right)^2\left[x_1^2(t)\tau + x_1^2(t)\tau\right] + \frac{\in a}{k}\left[x_1^2(t)\sigma + y_1^2(t)\sigma\right]\\
& + \frac{\in b}{k}\left[x_1^2(t)\sigma + z_1^2(t)\sigma\right] + \frac{\in a}{k}\left[y_1^2(t)\sigma + x_1^2(t)\tau\right]\\
& + a^2\left[y_1^2(t)\sigma + y_1^2(t)\sigma\right] + ab\left[y_1^2(t)\sigma + z_1^2(t)\sigma\right]\\
& + \frac{\in b}{k}\left[z_1^2(t)\tau + x_1^2(t)\tau\right] + ab\left[z_1^2(t)\sigma + y_1^2(t)\sigma\right]\\
& + b^2\left[z_1^2(t)\sigma + z_1^2(t)\sigma\right]\end{aligned}$$

Thus, we have:

$$\begin{aligned}\frac{dw_1(t)}{dt} \le & \, x_1^2(t)\frac{\in}{k}\left[\frac{-2}{x^*} + \frac{2\in\tau}{k} + (1+b)(\tau+\sigma)\right]\\
& + y_1^2(t)a\left[\frac{\in}{k}(\tau+\sigma) + 2\sigma(a+b)\right]\\
& + z_1^2(t)b\left[\frac{\in}{k}(\tau+\sigma) + 2\sigma(a+b)\right]\end{aligned}$$

$$-\frac{2a}{x^*}x_1(t)y_1(t)-\frac{2b}{x^*}x_1(t)z_1(t). \tag{A-4}$$

Next, we define w_{21} as in (3.11):

$$w_{21}(t)=\left[\frac{y_1(t)}{y^*}+\frac{1}{a}\int_{t-\sigma}^{t}x_1(s)ds\right]^2.$$

From (3.11) and (3.5) (ii) we have

$$\frac{dw_{21}(t)}{dt}=21\left[\frac{y_1(t)}{y^*}+\frac{1}{a}\int_{t-\sigma}^{t}x_1(s)ds\right]\left[-y_1(t)+\frac{1}{a}x_1(t)\right]$$

$$=\frac{-2y_1^2(t)}{y^*}+\frac{2}{a}\frac{1}{y^*}x_1(t)y_1(t)-\frac{2}{a}\int_{t-\sigma}^{t}y_1(t)x_1(s)ds+\frac{2}{a^2}\int_{t-\sigma}^{t}x_1(t)x_1(s)ds$$

$$\le\frac{-2y_1^2(t)}{y^*}+\frac{2}{a}\frac{1}{y^*}x_1(t)y_1(t)+\frac{1}{a}\left[y_1^2(t)\sigma+\int_{t-\sigma}^{t}x_1^2(s)ds\right]$$

$$+\frac{1}{a^2}\left[x_1^2(t)\sigma+\int_{t-\sigma}^{t}x_1^2(s)ds\right]. \tag{A-5}$$

Define $\quad w_2(t)=w_2(X)(t)=w_{21}(X)(t)+w_{22}(X)(t)$ (A-6)

Where $w_{22}(t)$ is defined so that the integral terms in (A-5) get eliminated. Thus

$$w_{22}(t)=\left(\frac{1}{a}+\frac{1}{a^2}\right)\int_{t-\sigma}^{t}\int_{s}^{t}x_1^2(l)dlds \tag{A-7}$$

From (A-5), (A-6) and (A-7), we get

$$\frac{dw_2(t)}{dt}\le x_1^2(t)\sigma\left[\frac{1}{a}+\frac{2}{a^2}\right]+y^2(t)\left[-\frac{2}{y^*}+\frac{\sigma}{a}\right]+\frac{2}{a}\frac{1}{y^*}x_1(t)y_1(t). \tag{A-8}$$

Similarly, we have:

$$w_3(t)=w_3(X)(t)=w_{31}(X)(t)+w_{32}(X)(t)$$

Where

$$w_{31}(t)=\left[\frac{z_1(t)}{z^*}+\frac{1}{b}\int_{t-\sigma}^{t}x_1(s)ds\right]^2$$

and $w_{32}(t)=\left(\frac{1}{b}+\frac{1}{b^2}\right)\int_{t-\sigma}^{t}\int_{s}^{t}x_1^2(l)dlds,$

and we get

$$\frac{dw_3(t)}{dt}\leq x_1^2(t)\sigma\left[\frac{1}{b}+\frac{2}{b^2}\right]+z_1^2(t)\left[\frac{-2}{z^*}+\frac{\sigma}{b}\right]+\frac{2}{b}\frac{1}{z^*}x_1(t)z_1(t) \qquad \text{(A-9)}$$

From (3.6) and using (A-4), (A-8) and (A-9) we get (3.16).

This completes the proof.

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A TWO WAREHOUSE DETERMINISTIC ECONOMIC ORDER QUANTITY MODEL WITH FINITE RATE OF REPLENISHMENT AND STOCK DEPENDENT

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ABSTRACT

A deterministic inventory model is developed for a single item with two warehouses one owned by the management (abbreviated as OW) and other is a rented warehouse (abbreviated as RW) by considering stock dependent consumption rate. It is assumed that storage capacity of OW is limited and that of RW is unlimited. The holding cost of RW is greater than OW and actual service of the customers is being done from OW, so it is economical to empty RW first. It is assumed that stocks are transferred from RW to OW according to K-Release Rule (KRR) in such a way that units in each transshipment is less than the capacity of OW. The model is set up by assuming finite replenishment rate and shortages are not allowed. Finally the solution of the model is discussed with the help of a numerical example.

Key words & Phrases : Demand, EOQ, Inventory, finite replenishment, L_2-system.
AMS Classification No. : 90-XX.

INTRODUCTION

Most of the inventory models are developed for a single warehouse. Due to large stock and limited capacity of existing storage facility *OW*, an additional storage facility *RW* which is located away from *OW* is hired to store excess items. In practice large stock attracts the management to either an attractive price discount for bulk purchase or the management resorts this practice where the acquisition costs are higher than the holding cost in *RW* or due to existing storage capacity limitations.

One warehouse, inventory storage facility is known as L_1- system whereas a two warehouse inventory storage facility is known as L_2- system. A good number of authors have considered L_2- system under different assumptions. Hartley [5]

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discussed this type of model neglecting the cost of transporting a unit from *RW* to *OW*. Sarma [11] extended it by considering the transportation cost of a unit from *RW* to *OW* with *K*-release rule and improved the working of the model [12, 13]. Dave [2] also improved the working of the model given by Sarma [11]. Murdeshwar and Sathe [7] extended the model given by Sarma [11] by considering finite production rate. Gupta and Vrat [4] developed a single level inventory model with stock dependent consumption rate. Mendus and Antony [8] developed a two warehouse inventory model by considering stock dependent consumption rate and infinite replenishment rate.

In this paper a L_2- system with stock dependent consumption rate with finite rate of replenishment with *K*- release rule (KRR) is considered, as large stock attracts customers particularly in consumer goods. Realistic assumptions given in this paper make the model more practical.

ASSUMPTIONS AND NOTATIONS

The following assumptions and notations are used for the development of the model.

(i) D is the demand rate and is taken as $D = \alpha + \lambda Q^{\delta}(\alpha, \lambda, \delta \text{ are constant})$.

(ii) Q is the highest inventory level.

(iii) The items of *RW* are transferred to *OW* in '*n*' shipment of which $K \leq W$ units are transported in each shipment.

(iv) Replenishment/Production rate is finite. Let '*R*' be the finite replenishment rate per time unit, which is known, constant and uniform during the period under consideration.

(v) Storage capacity of *OW* is W and that of *RW* is unlimited.

(vi) Let $\beta = 1 - D / R$ and $W_1 = W / \beta$.

(vii) Lead time is zero.

(viii) Set up cost is fixed and taken as A.

(ix) The transportation cost of K units from *RW* to *OW* is C_t at a time, which is constant over time.

(x) The holding cost/unit in *OW* is *H* and that in *RW* is *F* and *F*>*H*.

(xi) Shortages are not allowed.

(xii) Consumption takes place from *OW* only.

(xiii) *T* represents the total time period (cycle length).

MATHEMATICAL MODEL AND ANALYSIS FOR THE SYSTEM

We assume that a company purchases/products *Q* units, the finite rate of replenishment/production is *R*. During production stage the demand is satisfied from production and the excess quantities are transferred to warehouses as in Fig. (i). It is economical to fill *OW* first and when it is filled to its maximum capacity then excess production/replenishment is transferred to *RW*. When the production stops, the accumulated inventory can be used to satisfy further demand from *OW*. Due to this, inventory level in *OW* will fall from its maximum capacity *W*. When the stock level in *OW* reaches to (*W*-*K*) units, at this stage '*K*' units from *RW* are transported to *OW* to meet further demand and this process is repeated '*n*' times untill stock in *RW* is fully exhausted. The remaining (*W*-*K*) units in *OW* are used again at this stage. Also noting that after production stops, the inventory level in *OW* is '*W*' and that in *RW* is $z = (Q\beta - W)$ units. The inventory level is shown in figure (ii).

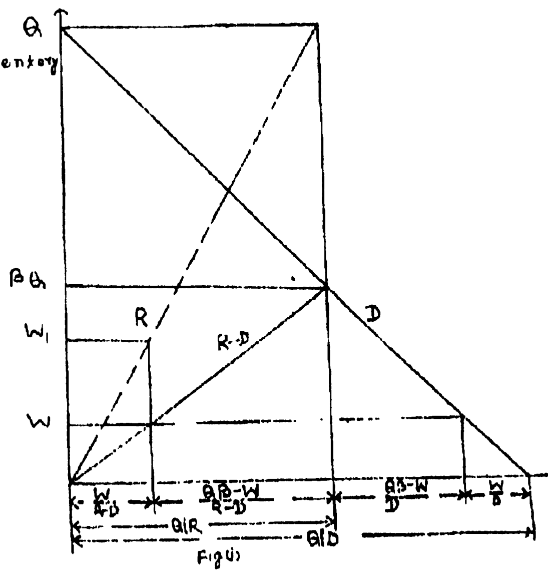

Fig (i)

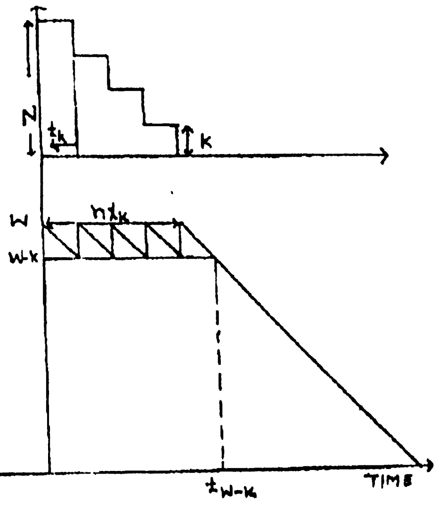

Fig (ii)

Case (i) (Before Consumption)

Initially, we have the total average cost as

$$C(Q)=\frac{AD}{Q}+\frac{HW}{2}+\frac{F(\beta Q-W)}{2} \tag{1}$$

Where $D=\alpha+\lambda Q^{\delta}$

For optimum value Q^{*} of Q, which minimize total average cost is obtained by solving

$$dC(Q)/dQ=O \text{ which gives } Q^{*}=\sqrt{\frac{2A\alpha}{\beta F}} \qquad (\text{If } \lambda=0)$$

As $\beta\to 1$ i.e. for infinite rate of replenishment

$Q^{*}=\sqrt{\dfrac{2A\alpha}{F}}$ which is same as Classical EOQ formula.

Case (ii) (During Consumption)

From figure (1). It is clear that during production stage.

Total inventory in $OW=\beta W_1 Q/R-\beta W_1^2/2R$

Total holding cost of these items = $H\left[\beta W_1 Q/R-\beta W_1^2/2R\right]$ (2)

Total inventory in $RW=\beta(Q-W_1)^2/2R$.

Total holding cost of these items = $F\beta(Q-W_1)^2/2R$. (3)

Further as soon as the production stops [as in Fig. (ii)]

there are W units in OW and $\beta(Q-W_1)$ units in RW.

Now using KRR, we have

Inventory units in RW are given by

$$= t_k[Z + (Z - K) + (Z - 2K) + \ldots\ldots\ldots + Z - (n-1)K]$$

$$= \frac{t_k(n+1)Z}{2}$$

where $t_k = K/D$, time taken for the consumption of K units.

Total holding cost of these items in $RW = F\beta k(Q - W_1)(n+1)/2D$ (4)

Cost of transporting K units from

RW to OW is given by $= nC_t = \beta(Q - W_1)C_t / K$ (5)

When K units are drawn from RW in each shipment more are carried in OW for a period of t_k and hence holding cost of these items is given by $KHt_k/2$.

Since there are 'n' such shipment and taking the initial K units into account, we have The total holding cost of these items

$$= \{(n+1)HKt_k\}/2 = \{(n+1)HK^2\}/2D \tag{6}$$

A quantity of (W-K) units is kept unused in OW for a period of $t_{w-k} = (n+1)t_k$ and during usage the average inventory in OW is given by (W-K)/2 units and for a period $(t - t_{w-k})$. Hence inventory holding cost in OW of these items is given by

$$H\{K(W-K)(n+1)/D + (W-K)^2/2D\} \tag{7}$$

The fixed order cost per order is A. Hence the total cost of inventory for the system using (2) to (7) is given by

$$C = A + H\{\beta W_1 Q/R - \beta W_1^2/2R\} + F\beta(Q - W_1)^2/2R + FK\beta(Q - W_1)(n+1)/2D$$

$$+ nCt + (n+1)HK^2/2D + H\{K(W-K)(n+1)/D + (W-K)^2/2D\}$$

Average inventory cost

$$C(Q,K) = C/T$$

Also we have $T = Q/D,\ Z = \beta(Q - W_t),\ n = Z/K$ and $D = \alpha + \lambda Q^{\delta}$,

$$C(Q,K) = A(\alpha + \lambda Q^{\delta})/Q + \beta FQ/2 + \beta W_1^2(F-H)/2Q - \beta KW_1(F-H)/2Q$$

$$+\beta K(F-H)/2 - \beta W_1(F-H) + \beta C_t(\alpha + \lambda Q^{\delta})(Q - W_1)/QK \tag{8}$$

For optimum value of Q and K, substituting $\delta C(Q,K)/\delta Q = 0$ and $\delta C(Q,K)/\delta K = 0$ $\delta C(Q,K)/\delta Q = O$ implies

$$Q^{\delta+1}\left[\frac{2C_t\lambda\beta}{K}\{\delta - W_1(\delta-1)\}\right] + 2A\lambda(\delta-1)Q^{\delta} + \beta FQ^2$$

$$= 2A\alpha + \beta W_1^2(F-H) - \beta KW_1(F-H) - 2C_t\beta W_1\alpha/K \tag{9}$$

which can not be solved for Q analytically and can be solved with the help of Newton Raphson method.

Also if $\lambda = O$

$$Q^{*} = \left[2A\alpha/\beta F + W_1^2(1 - H/F) - W_1K(1 - H/F) - 2C_tW_1\alpha/KF\right]^{\frac{1}{2}} \tag{10}$$

$\delta C(Q,K)/\delta K = O$ implies

$$K^{*} = \sqrt{\frac{2C_t(\alpha + \lambda Q^{\delta})}{F-H}}$$

$$\sqrt{\frac{2C_t\alpha}{F-H}} \qquad (\text{If } \lambda = 0) \tag{11}$$

Also from (11) and (10) we have

$$Q^{*}=\left[\frac{2A\alpha}{\beta F}+W_1^2(1-H/F)-\frac{2W_1}{F}\sqrt{2\alpha C_t(F-H)}\right]^{\frac{1}{2}}$$

COST REDUCTION DUE TO *K*-RELEASE RULE

The unit cost of KRR is $C_t' = C_t / K$. Suppose the unit cost of transportation C_t^* without bulk release rule. Bulk transportation is economical only if $C_t^* > C_t'$. Hence without KRR, we have the cost function

$$C(Q)=A(\alpha+\lambda Q^{\delta})/Q+\beta FQ/2+\beta W_1^2(F-H)/2Q-\beta W_1(F-H)$$

$$+\beta C_t^*(\alpha+\lambda Q^{\delta})-C_t^*\beta W_1(\alpha+\lambda Q^{\delta})/Q \qquad (13)$$

The optimum value Q' of Q which minimize (13) is obtained by solving

$dC(Q)/dQ=0$

This implies

$$\beta C_t^*\lambda\delta Q^{\delta+1}+Q^{\delta}\left[A\lambda(\delta-1)-C_t^*\beta W_1\lambda(\delta-1)\right]+\beta FQ^2=2A\alpha+\beta W_1^2(F-H)-2C_t^*\beta W_1\alpha$$

Equation (14) can not be solved for Q analytic and can be solved with the help of Newton Raphson method for obtaining optimal value of Q.

If $\lambda = 0$ then equation (14) reduces to

$$Q_1^1=\left[2A\alpha/\beta F+W_1^2(F-H)/F-2C_t^*W_1\alpha/F\right]^{\frac{1}{2}} \qquad (15)$$

KRR is economical if $\left[C(Q)-C(Q,K)\right]>0$

This implies $\left(C_t^*-C_t'\right)>(F-H)K/2\alpha$ (16)

must be satisfied.

Thus for a given situation inequality (16) is satisfied than KRR will be economical.

Relation between EOQ with and without KRR.

$$Q^* = \left[(Q')^2 + \frac{2C_t^* \alpha W_1}{F} - \frac{2W_1}{F} \sqrt{2\alpha C_k (F - H)} \right]^{\frac{1}{2}} \tag{17}$$

In general Q^* exceeds Q.

SPECIAL CASES

Case (i) If $\beta \to 1$ i.e. rate of replenishment is infinite.

Then equation (10) for optimum value of Q. becomes

$$Q^* = \left[2A\alpha / F + W^2 (1 - H / F) - WK(1 - H / F) - 2C_t W\alpha / KF \right]^{\frac{1}{2}} \tag{18}$$

Equation (18) is the same as equation (9) as that given by J. Mendus and Antony (8).

Case (ii) If Demand $D = \alpha$ and $\beta \to 1$.

Then equations (10), (11) and (12) of our Model reduces respectively to the equations (6) and (7) of Model (11).

Case (iii) When F is large, Q^* tends to W_1.

Further when $C_t \to 0$ Then $Q^* = \left[2A\alpha / \beta F + W_1^2 (1 - H / F) \right]^{\frac{1}{2}}$

In this case bulk transportation is not necessary.

NUMERICAL EXAMPLE

For a given inventory system, suppose we have $H = 2$, $F = 4$, $A = 100$, $W = 100$, $C_t = 2$, $C_t^* = 0.15$, $\alpha = 2000$, $\lambda = 0$, $R = 6000$, $D = 2000$ Then $\beta = 1 - D / R = 2/3$, $W_1 = W / \beta = 150$.

(a) With KRR, we have $K^* = 63, Q^* = 389$ and $C(Q, K) =$ Rs. 923.17.

(b) Without KRR. $Q = 372$; $C(Q) =$ Rs. 993.31

Also when $k = 63$, $C_t' = C_t / K = 0.0137$ and $C_t^* = 0.15$ clearly $C_t^* > C_t'$.

Also it is clear $Q^* > Q'$.

CONCLUDING REMARKS

In the present paper, we have formulated and solved a deterministic inventory model for two warehouses considering finite rate of replenishment and stock dependent consumption rate. Profitability of KRR has been desired. Also optimum batch size '*K*' is independent of production rate and capacity of own warehouse, but still from the example it is clear that KRR reduces cost considerably. Further it is worth remembering that one should consider the possibility of *RW* whenever lot size indicated by the classical EOQ model is greater than the capacity of own warehouse. This model further can be extended by considering shortages, as shortages is an unavoidable characteristic of most of the inventory systems implemented in practice.

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OSCILLATION THEOREMS FOR FIRST ORDER NONLINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS OF NEUTRAL TYPE

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INTRODUCTION

In this paper we are concerned with the oscillatory behaviour of first order neutral functional differential equation of the form

$$\frac{d}{dt}\left(x(t)+c(t)x[\tau(t)]\right)+\delta q(t)f\left(x[g(t)]\right)=0, \qquad (1.1;\delta)$$

where $\delta = \pm 1, \tau, c, g, q:[t_0,\infty) \dashrightarrow R = (t_0,\infty)$ and $f: R \rightarrow R$ are continuous. The following conditions are always assumed to hold:

(i) $\lim_{t\to\infty} g(t) = \infty$;

(ii) $\tau(t)$ is strictly increasing for $t \geq t_0 \geq 0$ and $\lim_{t\to\infty}\tau(t) = \infty$;

(iii) $xf(x) > 0$ for $x \neq 0$; and

(iv) $q(t) \geq 0$ and not identically zero on any ray of the form $[T,\infty)$ for some $T \geq t_0$.

By a solution of Eq. $(1.1;\delta)$ we mean a function $x:[T_x,\infty)\rightarrow R$ such that $x(t)+c(t)x[\tau(t)]$ is continuously differentiable and satisfies Eq. $(1.1;\delta)$ for all large $t \geq T_x$ The solutions which vanish for all large t will be excluded from our consideration.

A solution of Eq. $(1.1;\delta)$ is said to be oscillatory if it has an infinite sequence of zeros tending to infinity, otherwise, a solution is said to be nonoscillatory. Eq.$(1.1;\delta)$ is said to be oscillatory if all its solutions are oscillatory.

The oscillator behavior of first order functional differential equations of neutral type has been intensively studied in the literature in recent years. Most of the literature,

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however, is concerned with linear equations with constant coefficients and constant deviations for which oscillation criteria are given through analysis of the associated characteristic equations, and the references cited therein, and very little is known about the oscillation of Eq. $(1.1;\delta)$ whose coefficients and deviations are variable.

It seems that the known oscillation criteria fail to describe the oscillatory behavior of Eq. $(1.1;\delta)$ when f is not a monotonic function.

In this paper, we deal with the oscillation problem of nonlinear Eq. $(1.1;\delta)$ with variable coefficients and deviations and with f not a monotonic function. We establish some new sufficient conditions for the oscillation of all/and or bounded solutions of Eq. $(1.1;\delta)$ where the function $c(t)$ is either positive or negative for $t \geq t_0$. We also present some new sufficient conditions for the oscillation of the forced neutral equation.

$$(1.2;\delta) \qquad \frac{d}{dt}\left(x(t)+c(t)x[\tau(t)]\right)+\delta q(t)f\left(x[g(t)]\right)=e(t),$$

where $e:(t_0,\infty) \to R$ is continuous.

Examples are inserted in the text to illustrate the relevance of our theorems.

OSCILLATION OF EQ. (1.1; δ)

The following notations are needed in the sequel:

$$R_{t_0} = (-\infty,-t_0) \cup [t_0,\infty) \qquad \text{if} \quad t_0 > 0$$

$$= (-\infty,0) \cup (0,\infty) \qquad \text{if} \quad t_0 = 0$$

$C(R) = \{f: R \to R: f$ is continuous and $x f(x) > 0$ for $x \neq 0\}$.

$C_B(R_{t_0}) = \{f \varepsilon C(R): f$ is of bounded variation on any interval $[a,b] \subset R_{t_0}\}$

Suppose $t_0 > 0$ and $f \varepsilon C(R)$. Then $f \varepsilon C_B(R_{t_0})$ if and only if

$$f(x) = G(x)H(x) \text{ for all } x \in R_{t_0}$$

where

$G: R_{t_0} \rightarrow (0,\infty)$ is nondecreasing on $(-\infty,-t_0)$ and nonicreasing on $(t_{0,}\infty)$ and $\mathrm{H}: R_{t_0} \rightarrow \mathrm{R}$ and nondecreasing on R_{t_0}.

We assume that the functions c and τ are subject to one of the following conditions:

$$0 \le c(t) \le a_1 < 1 \qquad \tau(t) > t; \qquad \text{and} \tag{1.3}$$

$$1 \le b_1 \le c(t) \le b_2 \qquad \tau(t) < t; \tag{1.4}$$

$$0 \le c(t) \le a_1 < 1 \qquad \tau(t) < t; \qquad \text{and} \tag{1.5}$$

$$1 < b_1 \le c(t) \le b_2 \qquad \tau(t) < t. \tag{1.6}$$

We let

$$a = 1 - a_1 \quad \text{and} \quad b = \frac{b_1 - 1}{b_1 b_2}$$

The main results of this section now follow :

Theorem 1.1 Suppose that $f \varepsilon C(R_{t_0}), t_0 \ge 0$ and let the functions G and H be a pair of continuous components of F, H being the nondecreasing one. Then the following conclusions hold.

(I) Eq. (1.1;δ) is oscillatory, if condition (1.3) holds and for every positive constant $c<1$, $g(t) < t$ for $t \ge t_0$ and all large t, the first order delay equation

$$y'(t) + aq(t)G(c)H(y[g(t)]) = 0 \tag{1.7}$$

is oscillatory.

(II) Eq. (1.1;1) is oscillatory, if condition (1.4) holds and for every positive constant $c < 1, \tau^{-1} \circ g(t) < t$ for $t \ge t_0$ (where τ^{-1} denotes the inverse function of $\tau(t)$) and all large t, the first order delay equation

$$v'(t) + bq(t)G(c)H(v[\tau^{-1} \circ g(t)]) = 0 \tag{1.8}$$

is oscillatory.

Proof: Let $x(t)$ be a nonoscillatory solution of Eq. (1.1;1). We may assume that $x(t)$

is eventually positive, since a parallel argument holds if $x(t)$ is eventually negative. Define

$$y(t)=x(t)+c(t)x[\tau(t)], \tag{1.9}$$

then $y(t)>0$ for $t\geq t_0$ and Eq. (1.1;δ) implies that $y(t)$ is a decreasing function for all large t. Using this fact, we see from (1.9) that if condition (1.3) holds, then

$$\begin{aligned} x(t) &= y(t)-c(t)x[\tau(t)] \\ &= y(t)-c(t)[y[\tau(t)]-c[\tau(t)]x[\tau\circ\tau(t)]] \\ &\geq (1-c(t))y(t) \\ &\geq (1-a_1)y(t), \end{aligned}$$

or

$$x(t)\geq ay(t), \qquad \text{for all large } t, \tag{1.10}$$

and if condition (1.4) holds, then

$$\begin{aligned} x(t) &= \frac{1}{c[\tau^{-1}(t)]}\left(y[\tau^{-1}(t)]-x[\tau^{-1}(t)]\right) \\ &= \frac{y[\tau^{-1}(t)]}{c[\tau^{-1}(t)]}-\frac{1}{c[\tau^{-1}(t)]}\left(\frac{y[\tau^{-1}\circ\tau^{-1}(t)]}{c[\tau^{-1}\circ\tau^{-1}(t)]}-x\frac{[\tau^{-1}\circ\tau^{-1}(t)]}{c[\tau^{-1}\circ\tau^{-1}(t)]}\right) \\ &\geq \frac{c[\tau^{-1}\circ\tau^{-1}(t)]-1}{c[\tau^{-1}(t)]c[\tau^{-1}\circ\tau^{-1}(t)]}y[\tau^{-1}(t)] \\ &\geq \left(\frac{b_1-1}{b_1b_2}\right)y[\tau^{-1}(t)], \end{aligned}$$

or

$$x(t) \geq b \quad y[\tau^{-1}(t)], \text{ for all large } t. \tag{1.11}$$

(A) Suppose that condition (1.3) holds. Then from (1.10) and the fact that $y'(t) \leq 0$ for all large t, it follows that there exist a $T \geq t_0$ and a positive constant $c^* < 1$ such that for all $t \geq T$

$$x[g(t)] \leq y[g(t)] \leq c^* \tag{1.12}$$

and

$$x[g(t)] \geq ay[g(t)]. \tag{1.13}$$

Using (1.12) and (1.13) in Eq. (1.1:δ), we have

$$\begin{aligned} 0 &= y'(t) + q(t)f(x[g(t)]) \\ &= y'(t) + q(t)G(x[g(t)])H(x[g(t)]) \\ &\geq y'(t) + G(c^*)q(t)H(ay[g(t)]), \end{aligned}$$

or

$$w'(t) + aG(c^*)q(t)H(w[g(t)]) \leq 0 \text{ for } t \geq T, \tag{1.14}$$

where $w(t) = ay(t)$. It is easy to check that for all $t \geq T$

$$w(t) \geq aG(c)\int_t^\infty q(s)H(w[g(s)])ds.$$

The function $w(t)$ is positive and strictly decreasing for $t \geq T$.

(B) Suppose that condition (1.4) holds. Then from (1.11) and the fact that $y'(t) \leq 0$ for all large t, there exist a positive constant $c^* < 1$ and $aT^* \geq t_0$ such that (1.12) holds and

$$x[g(t)] \geq b \quad y[\tau^{-1} \circ g(t)] \qquad \text{for} \quad t \geq T^*. \tag{1.15}$$

Now, using (1.12) and (1.15) in Eq. (1.1; δ), we have,

$$y'(t) + G(c^*)q(t)H(by[\tau^{-1} \circ g(t)]) \leq 0 \text{ for } t \geq T^*,$$

or

$$w'(t) + bG(c^*)q(t)H\left(w\left[\tau^{-1} \circ g(t)\right]\right) \le 0 \text{ for } t \ge T^*. \tag{1.16}$$

Where $w(t) = by(t)$.

Proceeding as in the above case, we obtain the desired contradiction.
This completes the proof.

For any continuous function, $\sigma : [t_0, \infty) \to R$ with $\lim_{t \to \infty} \sigma(t) = \infty$, we define

$$R(\sigma) = \{ t \, \varepsilon [t_0, \infty] : t_0 \le \sigma(t) < 1 \}.$$

Also, we assume that

$$H(x) sgn \ge |x|^{\alpha} \text{ for } x > 0 \text{ and } \alpha > 0. \tag{1.17}$$

As an application of Theorem 1.1, we get the following oscillation criterion for Eq. (1.1;δ).

Corollary 1.1 Suppose that $f \, \varepsilon C(R_{t_0}), t_0 \ge 0$ and let the functions G and H form a pair of continuous components of F, H being the nondecreasing one. Then,

(I) Eq. (1.1;δ) is oscillatory if conditions (1.3) and (1.17) hold and one of the following conditions (A_1) or (B_1) holds:

(A1) $\alpha = 1$, and

$$g(t) < t \text{ and } g(t) \text{ is nondecreasing for } t \ge t_0; \tag{1.18}$$

and for every positive constant $c < 1$,

$$\lim_{\to \infty} inf \int_{g(t)}^{t} q(s) ds > \frac{1}{aeG(c)}. \tag{1.19}$$

(B_1) $a < 1$, and

$$\int_{R(g)} q(s) ds = \infty. \tag{1.20}$$

(II) Eq. (1.1;δ) is oscillatory if conditions (1.4) and (1.17) hold and one of the following conditions (C_1) or (D_1) holds :

(C_1) $\alpha = 1$, and

$\sigma(t) = \tau^{-1} \circ g(t) \le t$ and $\sigma(t)$ is nondecreasing for $t \ge t_0$ and (1.21)

for every positive constant $c<1$,

$$\liminf_{t\to\infty} \int_{\sigma(t)}^{t} q(s)ds > \frac{1}{beG(c)}. \tag{1.22}$$

(D_1) $\alpha < 1$ and

$$\int_{R(\tau^{-1}\circ g)} q(s)ds = \infty. \tag{1.23}$$

Proof: Let $x(t)$ be an eventually positive solution of Eq. $(1.1;\delta)$. As in the proof of Theorem 1.1, we obtain the inequalities (1.14) and (1.17). Now, we consider the inequality (1.14).

(A_1) From (1.17), inequality (1.14) takes the form

$$w'(t) + aG(c^*)q(t)w[g(t)] \le 0 \text{ for } t \ge T_1 \ge T. \tag{1.24}$$

In view of condition (1.19), it follows that inequality (1.24) cannot have an eventually positive solution. This contradicts the fact that $w(t) = a\,y(t) > 0$ eventually.

(B_1) Since $w[g(t)] \ge w(t)$ for all $t \in R(g)\cap[T,\infty)$, we have

$$\int_{R(g)\cap[T,\infty]} q(s)ds \le \frac{1}{aG(c^*)} \int_0^{w(T)} v^{-\alpha}dv$$

$$= \frac{1}{aG(c^*)}\left[\frac{w^{1-\alpha}(T)}{1-\alpha}\right] < \infty,$$

which contradicts (1.20)

Next, we consider the inequality (1.16). By a similar argument, as above, we obtain the desired conclusion.

The following example is illustrative.

Example 1.1 Each of the following neutral equations

$$\frac{d}{dt}\left(x(t)+\frac{1}{2}x[t+2\pi]\right)+\frac{3}{2}e^{\sin^2 t}\, e^{-x^{2}[t-3\pi]}x[t-3\pi]=0, t\geq 0, \tag{1.25}$$

and

$$\frac{d}{dt}(x(t)+2x[t-2\pi])+3e^{|\sin t|}e^{-|x[t-5\pi]|}x[t-5\pi]=0, t\geq 0, \tag{1.26}$$

has an oscillatory solution $x(t)=sint.$

All conditions of corollary 1.1 (I-A_1) and (II-C_1) are satisfied for Eq. (1.25) and Eq. (1.26) respectively.

BOUNDED OSCILLATION OF EQ. (1.1; δ)

In this section we are concerned with the oscillatory behavior of all bounded solutions of Eq. (1.1; δ) when condition (1.5) or (1.6) holds. Also we study the oscillatory behavior of Eq. (1.1) where c (t) is negative.

Theorem 1.2 Suppose that $f \in C\left(R_{t_0}\right), t_0 \geq 0$ and let the functions G and H form a pair of continuous components of F, H being the nondecreasing one. Then,

(I) All bounded solutions of Eq. (1.1;-1) are oscillatory if condition (1.5) holds and

one of the following conditions (A) or (B) is satisfied:

(A)

$$\int_0^{\infty} q(s)ds=\infty \tag{1.27}$$

(B) Condition (1.16) holds with $\alpha = 1$,

g (t) $\geq t$ and g (t) is nondecreasing for t $\geq t_0$, and for every (1.28)

positive constant c ≥ 1,

$$\liminf_{t\to\infty}\int_t^{g(t)} q(s)ds > \frac{1}{aeG(c)}. \tag{1.29}$$

(II) All bounded solutions of Eq. (1.1; δ) are oscillatory if condition (1.6) holds and one of the following conditions (C) or (D) is satisfied

(C) Condition (1.27) holds;

(D) Condition (1.17) holds with $\alpha = 1$,

$\sigma(t) = \tau^{-1} \circ g(t) \geq t$ and $\sigma(t)$ is nondecreasing for $t \geq t_0$, and for every positive constant $c \geq 1$, (1.30)

$$\liminf_{t \to \infty} \int_t^{\sigma(t)} q(s)ds > \frac{1}{beG(c)}. \tag{1.31}$$

Proof: Let $x(t)$ be a bounded and eventually positive solution of Eq. (1.1; δ).Define $y(t)$ by (1.9). Then $y(t)$ is eventually positive. From Eq. (1,1;δ) it follows that y (t) is nondecreasing for all large t, and it follows that (1.10) or (1.11) is satisfied according as condition (1.5) or (1.6) holds. Since $x(t)$ is bounded and $y(t)$ is nondecrasing for $t \geq t_0$, there exist positive constants $\beta_1 \geq 1$ and $\beta_2 > 0$ and a $T \geq t_0$ such that for $t \geq T$

$$x[g(t)] \leq \beta_1 \qquad \text{and} \qquad |y[g(t)]| > \beta_2 \tag{1.32}$$

and both (1.13) and (1.15) hold.

(I) Assume that condition (1.5) holds. Inview of (1.13) and (1.32), it follows from Eq. (1.1-1)

$$y'(t) \geq qG(\beta_1)H(a\beta_2) \qquad \text{for} \quad t \geq T. \tag{1.33}$$

Integrating (1.33) from T to t, we obtain

$$\infty > y(t) \geq y(T) + G(\beta_1)H(a\beta_2)\int_T^t q(s)ds$$

$$\to \infty \text{ as } t \to \infty,$$

a contradiction.

Next, using (1.17) with $\alpha=1$ and (1.32) in Eq. (1.1;δ) we have

$$y'(t) - aG(c_1)q(t)y[g(t)] \geq 0 \qquad \text{for} \quad t \geq T. \tag{1.34}$$

But, inview of condition (1.29), it follows that inequality (1.31) cannot have an eventually positive solution. This contradicts the fact that $y(t)$ is eventually positive.

(II) Assume that condition (1.5) holds. Inview of (1.15) and (1.32), it follows from Eq. (1.1;δ) that

$$y'(t) \geq G(\beta_1)q(t)H(by[\sigma(t)]) \qquad \text{for} \quad t \geq T.$$

Proceeding as in the proof of the above case, we obtain the desired conclusion. This completes the proof.

For illustration we consider the following example:

Example:1.2 Consider the neutral equations

$$\frac{d}{dt}\left(x(t)+\frac{1}{2}x[t-\sigma]\right)=\left(2+e^{-\sigma}\right)e^{t}\ cosh\,t^{2}\left(\frac{x\left[t^{2}\right]}{1+x^{2}\left[t^{2}\right]}\right)=0 \qquad t\geq 0, \qquad (1.35)$$

and

$$\frac{d}{dt}\left(x(t)+2x[t+\sigma]\right)=\left(2e^{-\sigma}+e^{-2\sigma}\right)\ cosh\ \left(e^{t+2\sigma}\right) \qquad (1.36)$$

$$sech\ \left(x[t+2\sigma]\right)x[t+2\sigma]=0, \qquad t\geq 0,$$

where σ is a positive constant, $\sigma \geq 2$.

All conditions of Theorem 1.2 (1-B) and(II-D) are satisfied for Eq. (1.35) and (1.36) respectively, and hence all bounded solutions of Eq. (1.35) and Eq. (1.36) are oscillatory.

We note that each of the equation (1.35) and (1.36) has an unbounded nonoscillatory solution $x(t)=e^{t}$.

Next, we present two criteria for the oscillation of all bounded solutions of Eq.(1.1;δ),$\delta=\pm 1$, when the function c (t) is assumed to be negative.

We assume that

$$0<-c(t)=\beta(t)<\beta_0 \text{ for some positive constant } \beta_0. \qquad (1.37)$$

Theorems. Let $f \varepsilon C\left(R_{t_0}\right)$, $t_0 \geq 0$ and conditions (1.18) and (1.37) hold and suppose that the functions G and H form a pair of continuous components of F,H being the nondecreasing one. Suppose that for every positive constant c and all large t the delay first orders equation

$$z'(t)+q(t)G(c)H\left(z[g(t)]\right)=0 \qquad (1.38)$$

is oscillatory, Then every bounded solution of Eq. (1:1;1) is oscillatory provided that either condition (1.27) is satisfied or in addition to conditions (1.17) with $\alpha = 1$ and (1.30) the condition

$$\liminf_{t\to\infty} \int_t^{\sigma(t)} q(s)ds > \frac{1}{e\beta_0 G(C)} \tag{1.39}$$

is satisfied.

Proof: Let $x(t)$ be a bounded and eventually positive solution of Eq. (1.1;1).

Define

$$y(t) = x(t) - \beta(t)x[\tau(t)]. \tag{1.40}$$

Then, inview of Eq. (1.1:δ).

$y'(t) = -q(t)f(x[g(t)]) \leq 0$ eventually and hence $y(t)$ is nonicreasing for all large. t Thus, $y(t)$ is of one sign for all large t, say $t \geq t_1 \geq t_0$.

We consider the following two cases:

Case 1: Let $y(t) > 0$ for $t \geq t_1$. There exist a positive constant γ and a $t_2 \geq t_1$ such that for all $t \geq t_2$.

$$x[g(t)] \leq \gamma \tag{1.41}$$

and

$$x[g(t)] \geq y[g(t)]. \tag{1.42}$$

Consequently Eq. (1.1;1) reduces to

$$y'(t) + G(\gamma)q(t)H(y[g(t)]) \leq 0 \text{ for } t \geq t_2.$$

Proceeding as in the proof of Theorem 1.1, we obtain the desired contradiction.

Case 2: Let $w(t) < 0$ for $t \geq t_1$. Using (1.41), we get

$$0 < -y(t) = u(t) = \beta(t)x[\tau(t)] - x(t)$$

$$\leq \beta(t)x[\tau(t)]$$

$$\leq \beta_0(t)x[\tau(t)] \qquad \text{for} \quad t \geq t_2$$

and here,

$$x(t) \geq \frac{1}{\beta_0} u[\tau^{-1}(t)] \text{ for } t \geq t_3 \geq t_2$$

From Eq. (1.1;δ) it follows that

$$w'(t) \geq \frac{1}{\beta_0} G(\gamma)q(t)H\left(w\left[\sigma^{-1} \circ g(t)\right]\right) \text{ for } t \geq t_3,$$

where u (t)=$\beta_0 w(t)$.

The rest of the proof is similar to that of Theorem 1.2 and hence is omitted. This completes the proof.

Theorem 1.4 Let $f \varepsilon$ $C(R_{t_0})$, $t_0 \geq 0$, conditions (1.21) and (1.37) hold, and let the functions G and H form a pair of continuous components of F, H being the nondecreasing one. If for every positive constant c and all large t the delay first order equation.

$$z'(t) + \frac{1}{\beta_0} G(c)q(t)H\left(z\left[\tau^{-1} \circ g(t)\right]\right) = 0 \tag{1.43}$$

is oscillatory, then every bounded solution of Eq. (1.1;-1) is oscillatory provided that either condition (1.27) holds, or in addition to condition (1.17) with $\alpha = 1$ and (1.28) the condition.

$$\liminf_{t \to \infty} \int_t^{g(t)} q(s)ds > \frac{1}{eG(c)} \text{ holds.} \tag{1.44}$$

Proof : Let x (t) be a bounded an eventually positive solution of Eq. (1.1;1). Define $y(t)$ by (1.40). Then, it follows that y (t) is nondecreasing and is of one sign for all large t. Now, we consider the following two cases:

Case 1: Suppose that y (t) >0 for $t \geq t_1 \geq t_0$. Then there exist a $t_2 \geq t_1$ and a positive constant γ such that

$x[g(t)] \leq \gamma$ and $x[g(t)] \geq w[g(t)]$, for $t \geq t_2$.

and hence

$$y'(t) \geq G(\gamma)q(t)H(y[g(t)]) \quad \text{for } t \geq t_2.$$

The rest of the proof is similar to that of Theorems 1.2 (I) and hence is omitted.

Case 2. Suppose that $y(t) < 0$ for $t \geq t_1 \geq t_0$. Set

$$0 > -y(t) = u(t)$$

and proceeds as in the proof of Theorem 1.3-case 2 to obtain

$$v'(t) + \frac{1}{\beta_0} G(\gamma)q(t)H(v[\tau^{-1} \circ g(t)]) \geq 0 \quad \text{for } t \geq t_2,$$

where $u(t) = \beta_0 v(t)$. We use an argument similar to the one used in the proof of Theorem 1.1 (II) to complete the proof.

The following examples are illustrative:

Example 1.3 The first order neutral equation

$$\frac{d}{dt}(x(t) - \gamma x[t + 2\pi]) + \left(\frac{1}{t^2} - \frac{\gamma}{(t + 2\pi)^2}\right)\left(\frac{1 + (t - \pi)^2}{t - \pi}\right) \tag{1.45}$$

$$\bullet \frac{x[t - x]}{1 + x^2[t - \pi]} = 0, \; t > \pi + y,$$

has a bounded nonoscillatory solution $x(t) = t$. Here γ is any positive constant. All conditions of Theorem 1.3 are satisfied the equation

$$\frac{d}{dt}y(t) + \left(\frac{1}{1 + c^2}\right)\left(\frac{1}{t^2} - \frac{\gamma}{(t + 2\pi)^2}\right)\left(\frac{1 + (t - \pi)^2}{t - \pi}\right)y(t - \pi) = 0,$$

for $\gamma > 0$ and $t > \pi$,

has a positive solution $y(t)$ with $\lim_{t \to \infty} y(t) = 0$ for every positive constant c.

Example 1.4 Consider the neutral equation

$$\frac{d}{dt}\left(x(t)-\gamma x\left[t^{2}\right]\right)+(2\gamma t-1)\left(t^{-\frac{1}{6}}+t^{\frac{5}{6}}\right)\left(\frac{x^{\frac{1}{3}}\left[\sqrt{t}\right]}{1+x^{2}\left[\sqrt{t}\right]}\right)=0, t>\frac{1}{2\gamma}, \qquad (1.46)$$

where γ is any positive constant. It is easy to check that for any positive constant c, the equation

$$y'(t)+\frac{1}{1+c^{2}}(2\gamma t-1)\left(t^{-\frac{1}{6}}+t^{\frac{5}{6}}\right)y^{\frac{1}{3}}\left[\sqrt{t}\right]=0$$

is oscillatory (see Corollary 1.1). Therefore, all conditions of Theorem 1.3 are satisfied and hence all bounded solutions of Eq. (1.46) are oscillatory. Note that Eq. (1.46) has an unbounded nonoscillatory solution $x(t)=t$.

Example 1.5 The first order neutral equation

$$\frac{d}{dt}\left(x(t)-2x[t-1]\right)=\left(\frac{2e-1}{e^{2}}\right)\left(1+e^{4}e^{-2t}\right)\left(\frac{x[t-2]}{1+x^{2}[t-2]}\right)=0, t>0 \qquad (1.47)$$

has a bounded nonoscillatory solution $x(t)=e^{-t}$. Here $\tau(t)=t-1, g(t)=t-2$ and hence $\tau^{-1}\circ g(t)=t-1>t$.

All conditions of Theorem 1.4 are satisfied, the delay equation,

$$y'(t)+\frac{1}{2\left(1+c^{2}\right)}\left(\frac{2e-1}{e^{2}}\right)\left(1+e^{4}e^{-2t}\right)y[t-1]=0, t>0$$

has a positive solution $y(t)$ with $\lim_{t\to\infty} y(t)=0$.

1.4. Oscillation of Eq. (1.2; δ)

In this section we establish some sufficient conditions under which Eq. (1.2; δ), $\delta=\pm 1$ is oscillatory.

We list the assumptions which are needed below :

$$0>c(t)=a_{1}>1, \tau(t)=t-\sigma, \quad \sigma \text{ is a positive constant;} \qquad (1.48)$$

$$c(t)=b_{1}=b_{2}>1, \tau(t)=t+\sigma, \quad \sigma \text{ is a positive constant.} \qquad (1.49)$$

Also, we assume the following conditions on the forcing term:

There exists a differentiable and oscillatory function

$\eta:[t_{0,}\infty]\rightarrow R$ Such that $\eta'(t)=e(t)$, (1.50)

and

$\eta(t)$ is periodic of period σ i.e. (1.51)

$\eta(t\pm\sigma)=\eta(t)$, where σ is defined as in (1.49) and (1.50).

Theorem 1.5. Let condition (1.50) hold, $c(t)=$ a, a is a nonnegative constant and $\tau(t)=t+\delta\sigma$, $\delta=\pm1$, and σ is a positive constant. If

(1.52) $\limsup_{t\to\infty}\eta(t)=+\infty$ and $\liminf_{t\to\infty}\eta(t)=-\infty$,

then,

(i) All bounded solutions of Eq. (1.2; δ) are oscillatory,

(ii) Eq. (1.2; δ) is oscillatory.

Proof: Let $x(t)$ be a bounded an eventually positive of Eq.(1.2; δ) and $x(t)$ be an eventually positive solution of Eq. (1.2; δ) Define

$y(t)=x(t)+ax[t+\delta_1\sigma]$ and $\tau(t)=y(t)-\eta(t)$, (1.53)

where $\delta_1=\pm1$.

Then Eq. (1.3;) take the form

$z'(t)=-\delta q(t)f(x[g(t)])$. (1.54)

Next, we consider the following two cases:

Case 1. $\delta_1=-1$. It follows that $z(t)$ is an increasing function, say on $[t_1\infty)$ for some $t_1\geq t_0\geq0$. We show that $z(t)>0$ for $t\geq T\geq t_1$.

If not, then $z(t)<0$ for $t\geq T\geq t_{1.}$ Hence

$$y(t)-\eta(t)>0 \text{ i.e } 0>y(t)<\eta(t) \text{ for } t\geq T,$$

which is a contradiction, since η (t) is oscillatory. Thus, we have

$$z(t) > 0 \text{ and } \quad z'(0) > 0 \quad \text{for} \quad t \geq T. \tag{1.55}$$

Taking *lim* superior in $y(t) > \eta(t)$ as $t \to \infty$, we obtain

$$\limsup_{t \to \infty} y(t) > \limsup_{t \to \infty} \eta(t) = \infty$$

which contradicts the fact that $y(t)$ is bounded.

Case 2. $\delta_1 = 1$**.** From (1.54), it follows that $z(t)$ is an eventually of one sign. As in the proof of case 1, above, we see that

$$z(t) > 0 \text{ and } z'(t) < 0 \qquad \text{for} \quad t \geq T. \tag{1.56}$$

Since $z(t) + \eta(t) = y(t) > 0$, we have

$z(t) > -\eta(t)$, from which it follows that

$$\limsup_{t \to \infty} z(t) \geq \limsup_{t \to \infty}(-\eta(t)) = -\lim_{t \to \infty} \mathit{ifn}\ \eta(t) = \infty$$

which contradicts the fact that $z(t)$ is bounded from above. This completes the proof.

For illustration we provide the following two examples:

Example 1.6 Consider the forced neutral equation

$$\frac{d}{dt}\left(x(t) + ax[t + \delta\sigma]\right) - \frac{\left(1 + ae^{-\delta\sigma}\right)e - t}{\left(1 - e^{-g(t)}\right)^{\gamma}}\left(\left(1 + \left(1 - e^{-g(t)}\right)^2\right)\right) \tag{1.57}$$

$$\bullet \frac{\left|x[g(t)]\right|^{\gamma}}{1 + x^2[g(t)]} sgn\, x[g(t)] = t.cost + sin\ t,\ t > 0,$$

where $\delta = \pm 1$, a and σ are nonnegative real numbers, α is a positive constant, g: $[t_0, \infty] \to R$ is continuous and $\lim_{t \to \infty} g(t) = \infty$. If we choose $\eta(t) = t.sint$, then all the hypotheses of Theorem 1.5 (i) are satisfied and hence every bounded solution of Eq. (1.57) is oscillatory, It is easy to verify that the corresponding unforced neutral equation, namely

$$\frac{d}{dt}\left(x(t) + ax[t + \delta\sigma]\right) - \frac{\left(1 + ae^{-\delta\sigma}\right)}{\left(1 - e^{-g(t)}\right)^{\alpha}}\left[1 + \left(1 - e^{-g(t)}\right)\right. \tag{1.58}$$

$$,\frac{|x[g(t)]|^{\alpha}}{1+x^{2}[g(t)]}\, sgn\, x[g(t)]=0, t>0,$$

has a bounded nonoscillatory solution $x(t)=1-e^{-t}$.

Example 1.7 Consider the forced neutral equation

$$\frac{d}{dt}(x(t)+ax[t+\delta\sigma])+\left(1+ae^{-\delta\sigma}\right)\left(e^{\alpha g(t)-t}\right) \tag{1.59}$$

$$\bullet\left(1+e^{-g(t)}\right)\left(\frac{|x[g(t)]|^{\alpha}}{1+|x[g(t)]|}\right) sgn\, x[g(t)]=e^{t}(sin\ t+ cost),$$

where $\delta=\underline{+}1$, a and σ are nonnegative constants, $\alpha>0$, and $g:(t_{0,}\infty)\rightarrow R$ is continuous and $\lim_{t\to\infty} g(t)=\infty$. Here, we choose $\eta(t)=e^{t}\ sin\ t$ and find that all the conditions of Theorem 1.5 (ii) are fulfiled. Thus, Eq. (1.59) is oscillatory we also note that the corresponding unforced neutral equation

$$\frac{d}{dt}(x+ax[t+\delta\sigma])+\left(1+ae^{-\delta\sigma}\right)\left(1+e^{-g(t)}\right) \tag{1.60}$$

$$\left(\frac{|x[g(t)]|^{\alpha}}{1+|x[g(t)]|}\right) sgn\ x[g(t)]=0, \qquad t>0,$$

has a nonoscillatory solution $x(t)=e^{-t}$.

Theorem 1.6 Let $f\ \varepsilon\ C\left(R_{t_0}\right), t_0\geq 0$ and conditions (1.50) and (1.51) hold.

Suppose that G and H form a pair of continuous components of F, H being the nondecreasing one. Then the following conclusions hold:

(I) All bounded solutions of Eq. (1.2; δ) are oscillatory if condition (1.48) holds and either

(A) Condition (1.27) holds; or

(B) In addition to condition (1.7) with $\alpha=1$ and (1.28) the condition (1.29) holds for every constant $c\geq 1$

(II) All bounded solutions of Eq. (1.2; δ) are oscillatory if condition (1.49) holds and either:

(C) Condition (1.27) holds: or

(D) In addition to conditions (1.17) with $\alpha = 1$ and (1.30) the condition (1.31) holds for every constant $c \geq 1$.

Proof: Let $x(t)$ be a bounded and eventually positive solution of Eq. (1.2; δ). As in the proof of Theorem 1.5, we define the functions $y(t)$ *and* $z(t)$ as in (1.53) and obtain

$$z(t) > 0 \quad \text{and} \quad z'(t) > 0 \text{ eventually.} \tag{1.61}$$

Next, we consider:

(I) Let condition (1.48) hold. From (1.50), (1.51), (1.53) with $\delta_1 = -1$ and

(1.61), we see that

$$\begin{aligned} x(t) &= z(t) + \eta(t) - a_1 x[t-\sigma] \\ &= z(t) + \eta(t) - a_1[z(t-\sigma) + \eta[t-\sigma] - a_1 x[t-2\sigma]] \\ &\geq a(z(t) + \eta(t)) \text{ for all large } t, \end{aligned}$$

where $a = 1 - a_1$. Next by (1.51), there exists a $T \geq t_0$ *(say)* such that for all $t \geq T$

$$x(t) \geq a(z(t) + \eta(T)) = w(t).$$

Clearly $az'(t) = w'(t)$ and $w(t) > 0$ for all $t \geq T$. Since $x(t)$ is bounded, there exist *a* $T_1 \geq T$ and a positive constant γ such that for $t \geq T_1$

$$x[g[t]] \leq \gamma \quad \text{and} \quad x[g(t)] \geq w[g(t)]. \tag{1.62}$$

Thus, $\qquad w'(t) \geq aq(t)G(\gamma)H(w[g(t)])$ for all $t \geq T_1$.

The rest of the proof is similar to the proof of Theorem 1.2 (I) and hence is omitted.

(II) Let condition (1.49) hold. From (1.50), (1.53) with $\delta_1 = 1$ and (1.61), we obtain

$$x(t) = \frac{1}{b_1}(z[t-\sigma] + \eta[t-\sigma] - x[t-\sigma])$$

$$= \frac{1}{b_1}(z[t-\sigma]+\eta[t-\sigma]) - \frac{1}{b_1^2}(z[t-2\sigma]+\eta[t-2\sigma]-x[t-2\sigma])$$

$$\geq \left(\frac{b_1-1}{b_1^2}\right)(z[t-\sigma]+\eta[t-\sigma])$$

$$= b(z[t-\sigma]+\eta[t-\sigma]) \text{ for all large } t,$$

where $b = \dfrac{b_1-1}{b_1^2}$.

Once again by (1.51), there exists a $T^* \geq t_1 + \sigma$ for some $t_1 \geq t_0$ Such that for $t \geq T^*$ we have

$$x(t) \geq b\left(z[t-\sigma]+\eta\left[T^*-\sigma\right]\right) = v[t-\sigma],$$

and $bz'(t) = v'(t)$ and $v(t) > 0$ for $t \geq T^*$. Since $x(t)$ is bounded there exist a constant $\gamma_1 \geq 1$ and a $T_1^* \geq T^*$ such that

$$x[g(t)] \leq \gamma_1 \text{ and } x[g(t)] \geq v[g(t)-\sigma] \text{ for all } t \geq T_1^* \tag{1.63}$$

Thus $$v'(t) \geq bG(\gamma_1)q(t)H(v[g(t)-\sigma]) \text{ for } t \geq T_1^*.$$

Proceeding as in the proof of Theorem 1.2 (II), we obtain the desired conclusion and the proof is now complete.

For illustration we consider the following two examples:

Example 1.8 Consider the forced neutral equation

$$\frac{d}{dt}\left(x(t)+cx[t-2\pi\delta_1]\right) = ce^{|\cos t|}\left(e^{-|x[g(t)]|}\right)$$

$$\left(|x[q(t)]|^{\alpha}\right) sgn\ x[g(t)] + cos t,\ t>0 \tag{1.64;δ_1}$$

where $\alpha > 0, \delta_1 = \pm 1$ and c is a positive constant such that

(i) $\delta_1 = 1$ if $0<c<1$; *and*

(ii) $\delta_1 = -1$ if $c>1$,

$g:[t_0,\infty) \to R$ is continuous and $\lim_{t\to\infty} g(t) = \infty$.

Here, we choose the functions, g, H and η as follows

$$G(x) = e^{-|x|}, H(x) = |x|^{\alpha}\, sgn\ x \text{ and } \eta(t) = Sin\ t.$$

Then $\eta(t)$ is oscillaory and periodic of period 2π. Also,

$$\eta'(t) = cost = e(t).$$

All conditions of Theorem 1.6 (I-A) and (II-C) are satisfied for (i) and (ii) respectively and hence all bounded solutions of Eq. $(1.64;\delta_1)$ are oscillatory.

We observe that if $\alpha = 1$ and $g(t) = t - \frac{3\pi}{2}$, Eq. $(1.64;\delta_1)$ has a bounded and oscillatory solution $x(t) = sint$.

Example. 1.9 Consider the forced neutral equation

$$\frac{d}{dt}\left(x(t) + cx[t - 2\pi\delta_1]\right) = \left(\frac{1 + c + c\cos t}{\tanh(t + \sin t)}\right) \qquad (1.65;\delta)$$

$$sech\left(x[g(t)]\right)\sinh\left(x[g(t)]\right) + \cos t,\ t > 0$$

where $\delta_1 = \pm 1$, c and g are defined as in Eq. $(1.164;\delta_1)$ and (i) and (ii) in Example 1.8 hold. Here, we let

$G(x) = sech\ x$, $H(x) = sinh\ x$ and $\eta(t) = sint$.

All conditions of Theorem 1.6 (I.A.) and (II.C) are satisfied for (i) and (ii) hold respectively, and hence every bounded solution of Eq. $(1.65;\delta)$ is oscillatory, It is easy to check that if $g(t) = t$, Eq.$(1.6;\delta_1)$ has an unbounded nonoscillatory solution x $x(t) = t + sint$, $t > 0$.

SOME GENERAL REMARKS

1. We note that if $f(x)$ in Eq. $(1.;\delta_1)$ and $(1.2;\delta_1)$, $\delta_1 = \pm 1$, is a nondecreasing function for $x \neq 0$ (i.e.,the components G (x) of $F(x)$ is identically 1), then the boundedness requirement on the solution of Eq. $(1.1.\delta)$ and Eq. $(1.2;\delta)$ in Theorems 1.2, 1.3, 1.4 and 1.6 can be removed.

2. The function $f(x)$ in Eq. $(1.2;\delta)$, $\delta_1 = \pm 1$ may not satisfy the linearization conditions. For example, we see that if

$$\frac{f(x)}{x} = \frac{|x|^{\alpha-1}}{1 + x^2} \to 1 \quad \text{as} \quad x \to 0, \quad \alpha = 1$$

$$\to 1 \quad \text{as} \quad x \to \pm\infty, \quad \alpha = 3$$

$$\to \infty \quad \text{as} \quad x \to \pm\infty, \quad \alpha = 3\cdot$$

The lineariztion conditions are satisfied while for the case when

$$\frac{f(x)}{x} = e^{-|x|}|x|^{\alpha-1} \to 0 \text{ as } x \to 0 \text{ and } x \to \pm\infty, \quad \alpha > 1.$$

The linearization conditions fail. Thus, the results of this paper are applicable to a larger class of equations of type $(1.1;\delta_1)$ and $(1.2;\delta_1)$, $\delta_1 = \pm 1$.

3. The forcing term $e(t)$ in Eq. $(1.1;\delta_1)$ and the function η defined by (1.50) and (1.51) are assumed to be oscillatory and periodic and hence bounded for $t \geq t_0$. We note that:
(i) The forcing term e considered in Eq. $(1.2;\delta_1)$ is not necessarily small.
(ii) From Examples 1.6 and 1.7 it is evident that the presence of a forcing term can generate oscillations in an otherwise nonoscillatory equation.

Finally, it should be interesting to obtain results similar to Theorems 1.2-1.6 for Eq.(1.1;-1) and Eq. (1.2;-1) without the boundedness restriction on the solutions under consideration.

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PROPAGATION OF STRONG SPHERICAL DIVERGING SHOCK WAVE AND CHANGE IN ENTROPY OF NON-UNIFORM MEDIUM

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ABSTRACT

Chester-Chisnell-Whitham method has been used to study the propagation of shperical diverging strong shock wave in non-uniform medium. Assuming an initial density distribution law as $\rho_0 = \rho' r^{-\omega}$ where ρ' is the density at the centre and ω is a constant, analytical relation for shock velocity has been derived. The expressions for pressure, particle velocity, sound velocity, temperature and change in entropy of the medium immediately behind the shock have been obtained. Finally, the results accomplished here have also been compared with earlier results as well as with those for converging shock. It is found that change in entropy increases as shock propagates in the medium. This agrees with the universal law of increase of entropy.

Key words : Diverging shock wave (Non-uniform medium), Change in entropy

INTRODUCTION

On account of its immense importance in Astrophysics and Thermodynamics, propagation of shock wave (plane, cylindrical, and spherical) has received considerable attention of many workers [1-10]. Considering the effect of overtaking disturbances, the propagation of strong diverging shock in non-uniform medium, the temperature variation has been computed by Yadav and Rana [11].

Yadav *et al* [12-13] have studied the propagation of spherical converging strong shock in uniform medium and obtained the change in entropy and temperature of non-uniform medium on propagation of spherical converging strong shock wave.

Recently, Yadav and Gangwar [14] have studied the propagation of spherical converging strong shock in non-uniform medium. In the present paper CCW method has been used to study the propagation of spherical diverging strong

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shock in non-uniform medium. Assuming an initial density distribution law as $\rho = \rho' r^{-\omega}$ where ρ' is the density at the centre and ω is a constant, analytical relations for the shock velocity, the pressure, the particle velocity, the sound velocity, the temperature and the change in entgropy of medium immediately behind the shock have been derived and their variations with propagation distance (r), constant (ω) and adiabatic index (γ) have been given through figures.

Finally, the dependence of these parameters on adiabatic index (γ), propagation distance (r) and constant (ω) have been discussed.

It is found that temperature and change in entropy of the medium increase as shock propagates. This agrees with earlier results [8, 13] as well as universal law of increase of entropy. For $\omega > \frac{4\gamma}{\left\{2+\sqrt{2\gamma(\gamma-1)}\right\}}$, the cange in entropy increase with ω and decrease with adiabatic index γ and the change in entropy is reverse for $\omega < \frac{4\gamma}{\left\{2+\sqrt{2\gamma(\gamma-1)}\right\}}$.

THEORY

The equations governing the spherically symmetrical flow enclosed by the shock front are :

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{1}{\rho}\frac{\partial p}{\partial r} = 0 \tag{1}$$

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial r}\right)\rho + \rho\left(\frac{\partial u}{\partial r} + \frac{2u}{r}\right) = 0 \tag{2}$$

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial r}\right)\left(p\rho^{-\gamma}\right) = 0 \tag{3}$$

where, $u(r,t)$, $p(r,t)$ and $\rho(r,t)$ denote respectively the particle velocity, the pressure and density at a distance r from the origin at time t, and γ is the adiabatic index of gas.

Let p_0 and ρ_0 denote the undisturbed value of pressure and density in front of shock and p_1, ρ_1 and u_1 be the values of respective quantities at any point immediately after passage of shock, the Rankine-Hugoniot conditions are :

$$p_1 = \rho_0 a_0^2 \left[\frac{2M^2}{\gamma+1} - \frac{\gamma-1}{\gamma(\gamma+1)} \right] \tag{4}$$

$$p_1 = \rho_0 \frac{(\gamma+1)M^2}{(\gamma-1)M^2+2}$$

$$u_1 = \frac{2a_0}{\gamma+1}\left(M - \frac{1}{M} \right)$$

$$a = a_0 \frac{\sqrt{\left[2\gamma M^2 - (\gamma-1)\right]\left[(\gamma-1)M^2+2\right]}}{(\gamma+1)M}$$

Where $M = U / a_0$ is Mach number, U is the shock velocity, a and a_0 are the sound velocities in disturbed and undisturbed medium respectively.

The temperature and change in entropy are given in MaGraw-Hill Encyclopedia of science and technology [3]

$$\frac{T}{T_0} = \frac{\left[2\gamma M^2 - (\gamma-1)\right]\left[2+(\gamma-1)M^2\right]}{(\gamma+1)^2 M^2} \tag{5a}$$

$$\frac{\Delta S}{R} = \frac{1}{\gamma-1} \text{In} \frac{\left[2\gamma M^2 - (\gamma-1)\right]\left[2+(\gamma-1)M^2\right]^{\gamma}}{(\gamma+1)^{\gamma+1} M^{2\gamma}} \tag{5b}$$

Where T and T_0 are the temperature of perturbed and unperturbed media respectively, ΔS is the change in entropy between perturbed and unperturbed media and R is the universal gas constant.

For strong shock i.e. $M>>1$, conditions (4) and (5) reduce to

$$p=\frac{2\rho_0 U^2}{\gamma-1},\ \rho=\frac{\gamma+1}{\gamma-1}\rho_0,\ u=\frac{2U}{\gamma+1},\ a=\frac{\sqrt{2\gamma(\gamma-1)}}{\gamma+1}U \tag{6}$$

$$\frac{T}{T_0}=\frac{2\gamma(\gamma-1)}{(\gamma+1)^2}\left(\frac{U}{a_0}\right)^2 \tag{7a}$$

$$\frac{\Delta S}{R}=\frac{1}{\gamma-1}\mathrm{In}\left\{\frac{2\gamma(\gamma-1)^\gamma}{(\gamma+1)^{\gamma+1}}\right\}\left(\frac{U}{a_0}\right)^2 \tag{7b}$$

For spherical diverging shocks, the characteristic form of the system of equations (1)-(3), i.e. the form in which each equation contains derivatives in only one direction in (r, t) plane, is

$$dp+\rho a du+\frac{2\rho a^2 u}{(u+a)}\frac{dr}{r}=0 \tag{8}$$

It is assumed that the medium in which shock propagates is non-uniform, whose initial density distribution is given by

$$\rho_0=\rho' r^{-\omega} \tag{9}$$

where ρ' is the density at the centre and ω is a constant.

Substituting conditions (6) into equation (8), we get

$$\frac{dU}{U}+\frac{1}{2+\sqrt{2\gamma/(\gamma-1)}}\left[-\omega+\frac{4\gamma}{2+\sqrt{2\gamma(\gamma-1)}}\right]\frac{dr}{r}=0 \tag{10}$$

On integrating equation (10), expression for shock velocity can be written as

$$U=Kr^{\left[\frac{1}{2+\sqrt{2\gamma/(\gamma-1)}}\left\{\omega-\frac{4\gamma}{2+\sqrt{2\gamma(\gamma-1)}}\right\}\right]} \tag{11}$$

where K is the constant of integration.

The expression for non-dimensional shock velocity, there after called as shock strength can be written as

$$\frac{U}{a_0} = A\gamma^{-1/2} r^{\left[\frac{1}{2+\sqrt{2\gamma/(\gamma-1)}}\left\{\omega - \frac{4\gamma}{2+\sqrt{2\gamma(\gamma-1)}}\right\}\right]} \tag{12}$$

where $A = k\sqrt{\rho_0 / p_0}$

From equation (6), (7) and (12), the pressure, the particle velocity, the sound velocity, the temperature and the change in entropy are given by expressions (13)-(17) respectively.

$$\frac{p}{p_0} = \frac{2A^2}{(\gamma+1)} r^{\left[\frac{2}{2+\sqrt{2\gamma/(\gamma-1)}}\left\{\omega - \frac{4\gamma}{2+\sqrt{2\gamma(\gamma-1)}}\right\}\right]}, \tag{13}$$

$$\frac{u}{a_0} = \frac{2A}{(\gamma+1)\sqrt{\gamma}} r^{\left[\frac{1}{2+\sqrt{2\gamma/(\gamma-1)}}\left\{\omega - \frac{4\gamma}{2+\sqrt{2\gamma(\gamma-1)}}\right\}\right]} \tag{14}$$

$$\frac{a}{a_0} = \frac{A\sqrt{2(\gamma-1)}}{(\gamma+1)} r^{\left[\frac{1}{2+\sqrt{2\gamma/(\gamma-1)}}\left\{\omega - \frac{4\gamma}{2+\sqrt{2\gamma(\gamma-1)}}\right\}\right]} \tag{15}$$

$$\frac{T}{T_0} = \frac{2A^2(\gamma-1)}{(\gamma+1)^2} r^{\left[\frac{2}{2+\sqrt{2\gamma/(\gamma-1)}}\left\{\omega - \frac{4\gamma}{2+\sqrt{2\gamma(\gamma-1)}}\right\}\right]} \tag{16}$$

$$\frac{\Delta S}{R} = \frac{1}{(\gamma-1)} In \frac{2A^2(\gamma-1)^{\gamma}}{(\gamma+1)^{\gamma+1}} r^{\left[\frac{2}{2+\sqrt{2\gamma/(\gamma-1)}}\left\{\omega - \frac{4\gamma}{2+\sqrt{2\gamma(\gamma-1)}}\right\}\right]} \tag{17}$$

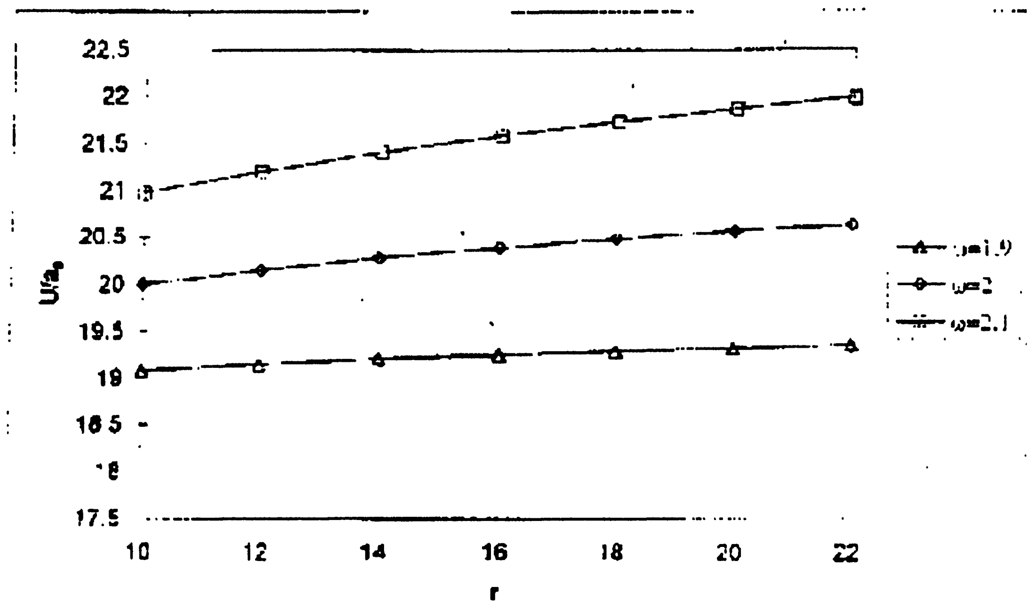

Figure 1 Variation of shock velocity (U/a_0) with propagation distance (r) for adiabatic index γ=1.33

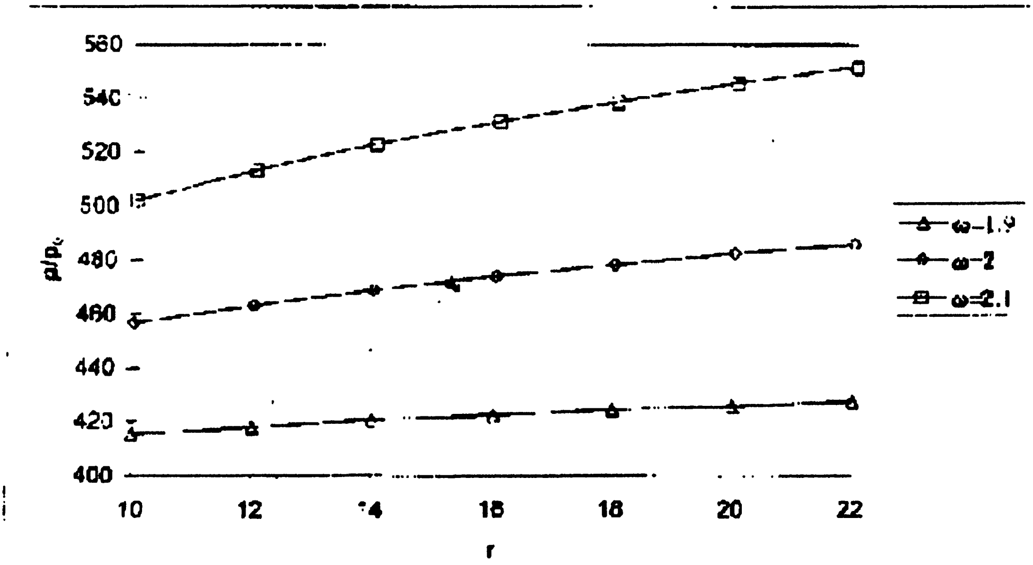

Figure 2 Variation of pressure (p/p_0) with propagation distance (r) for adiabatic index γ=1.33

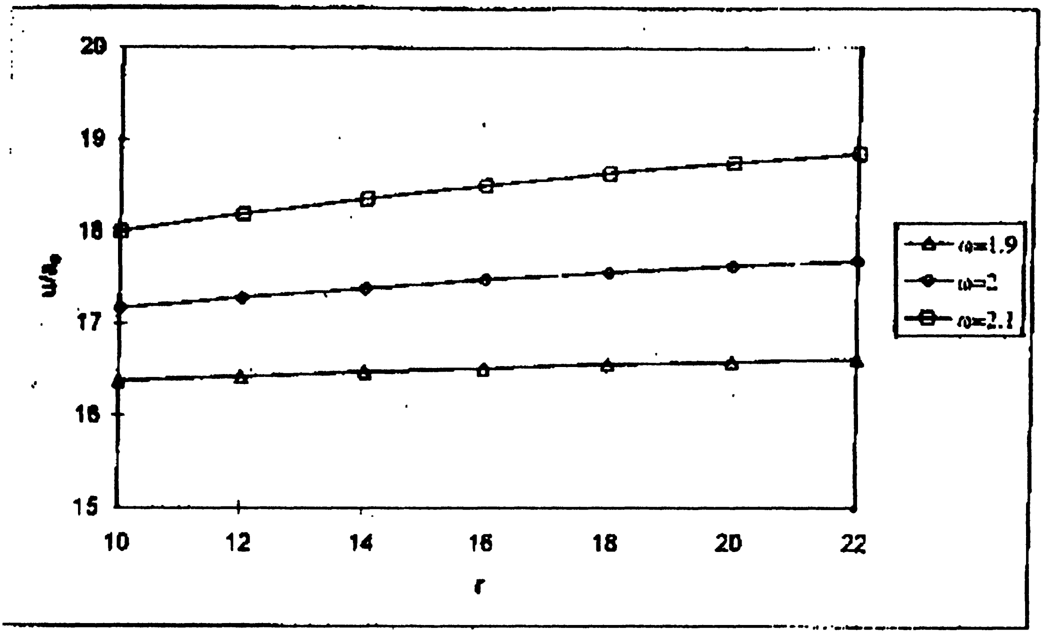

Figure 3 Variation of particle velocity (u/a_0) with propagation distance (r) for adiabatic index $\gamma=1.33$

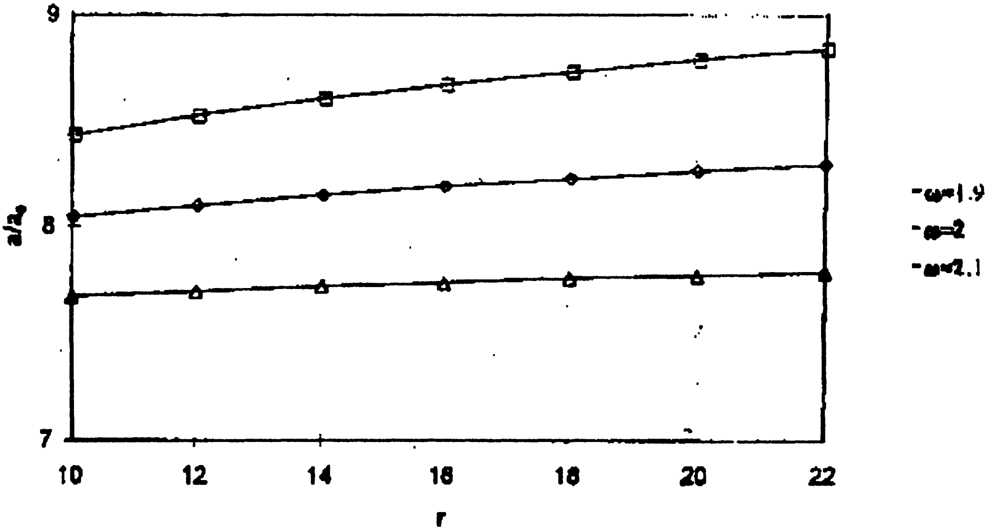

Figure 4 Variation of sound velocity (a/a_0) with propagation distance (r) for adiabatic index $\gamma=1.33$

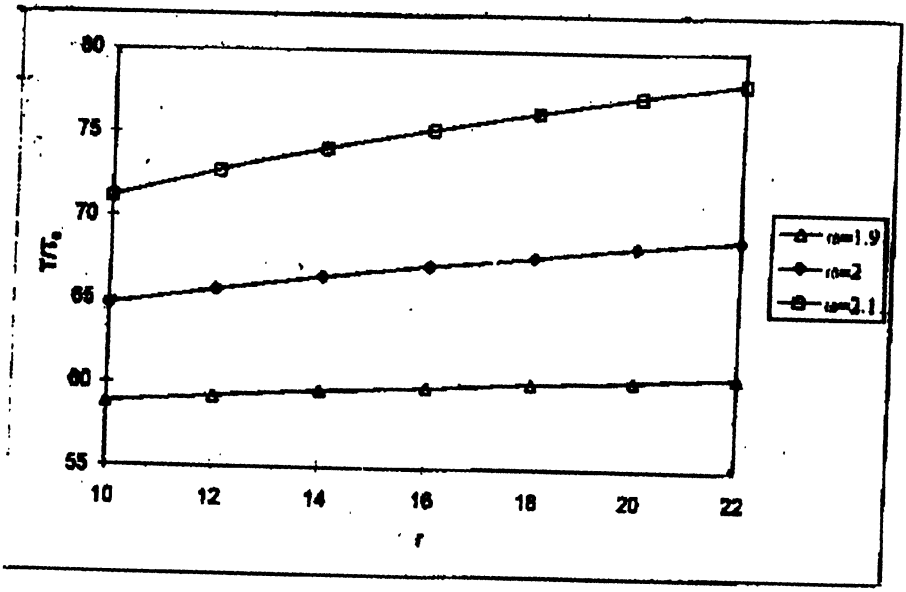

Fig 5 Variation of temperature (T/T_0) with propagation distance (r) for adiabatic index $\gamma=1.33$

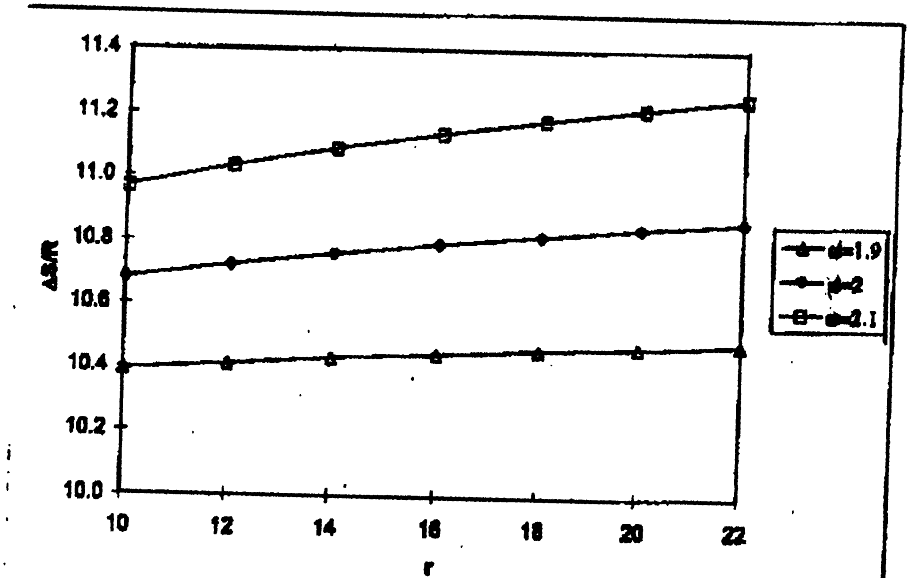

Fig 6 Variation of change in entropy ($\Delta S/R$) with propagation distance (r) for adiabatic index $\gamma=1.33$

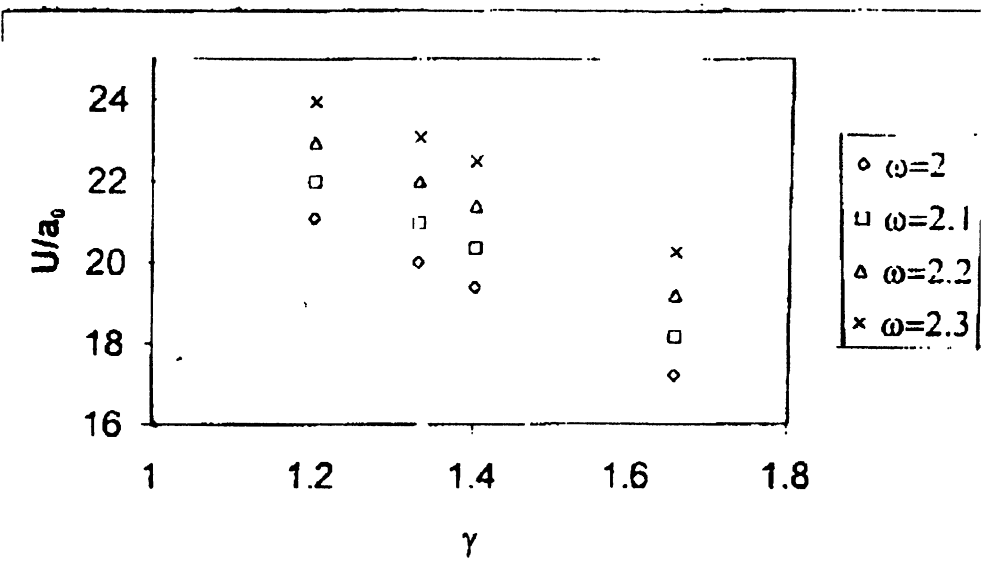

Fig 7 Variation of shock velocity (U/a_0) with adiabatic index γ at propagation distance r = 10

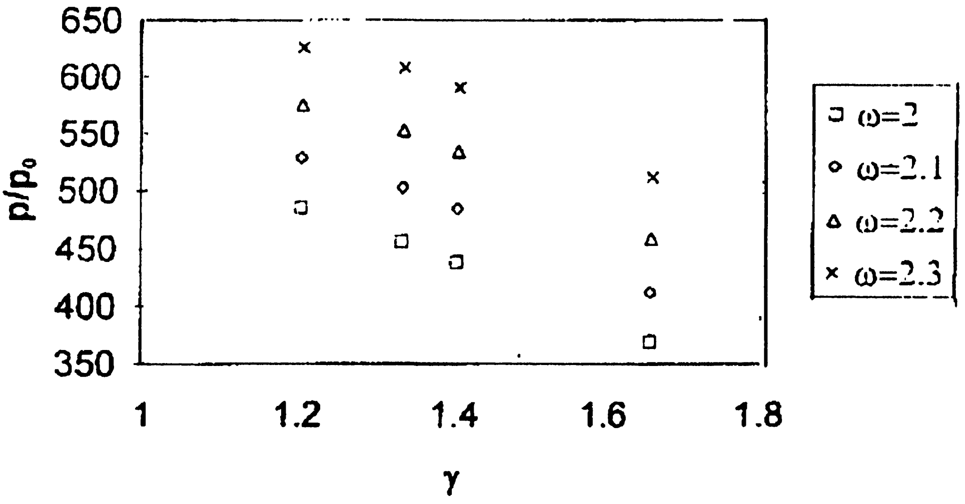

Fig 8 Variation of pressure (p/p_0) with adiabatic index γ at propagation distance r = 10

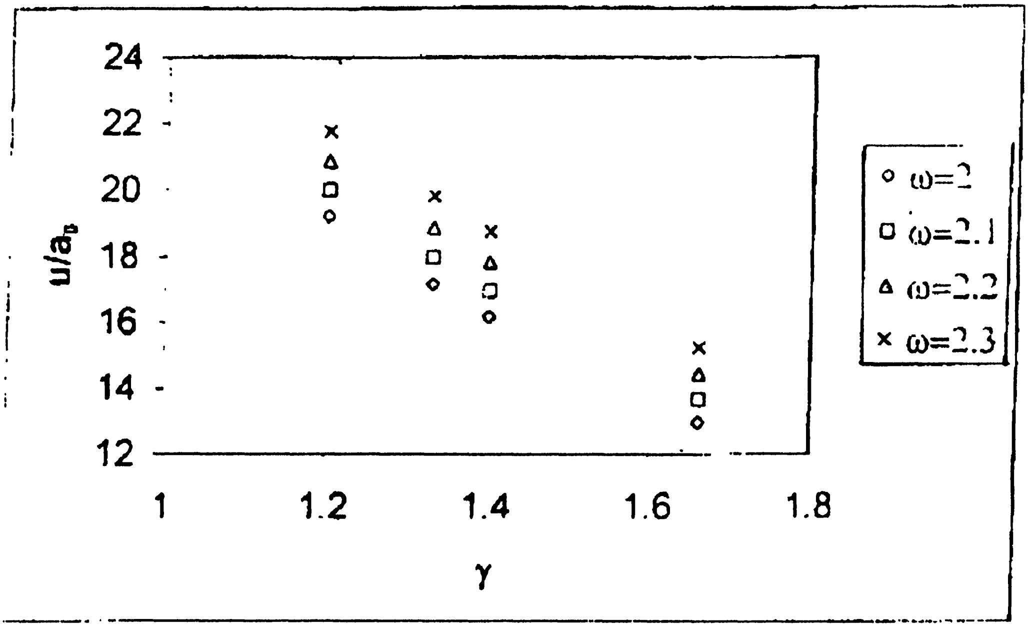

Fig 9 Variation of particle velocity (u/a₀) with adiabatic index γ at propagation distance r = 10

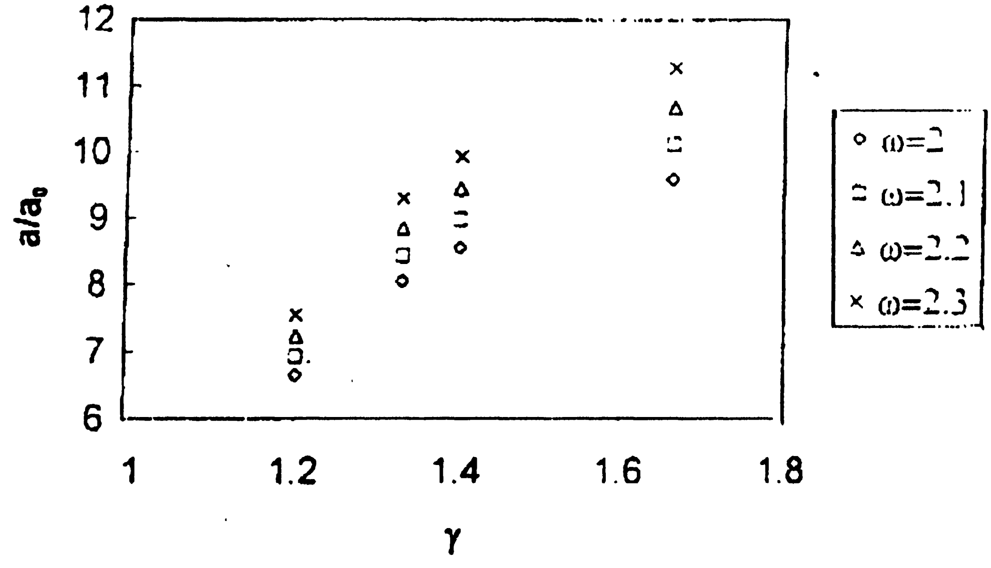

Fig 10 Variation of sound velocity (a/a₀) with adiabatic index γ at propagation distance r = 10

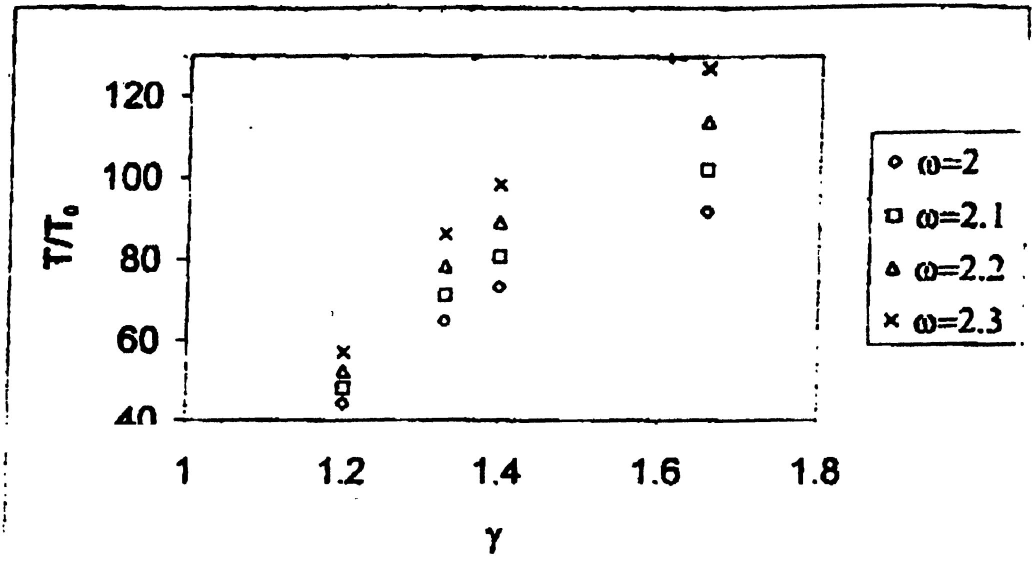

Fig 11 Variation of temperature (T/T_0) with adiabatic index γ at propagation distance r = 10

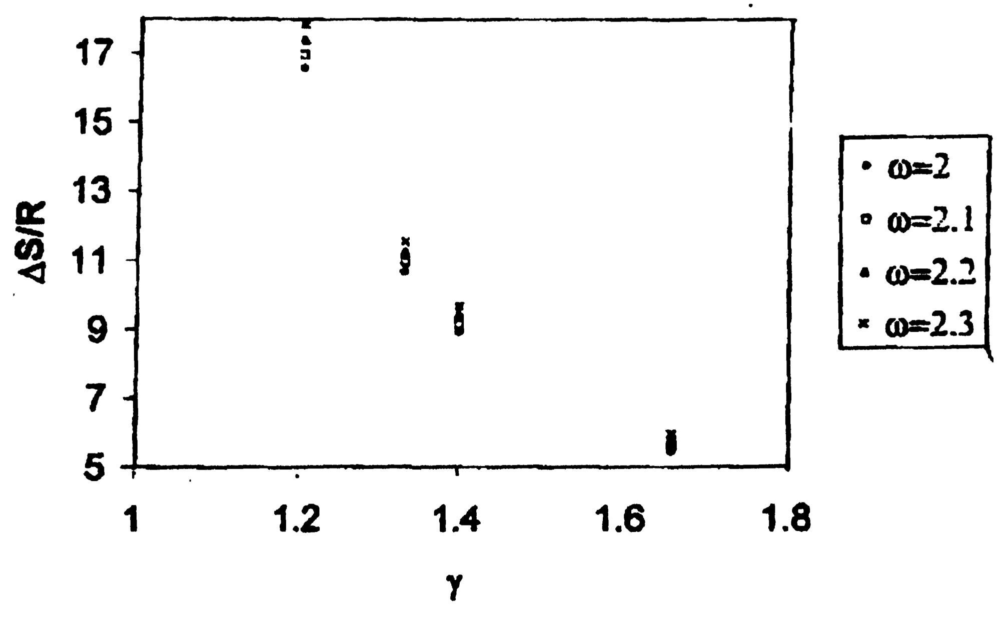

Fig 12 Variation of change in entropy ($\Delta S/R$) with adiabatic index γ at propagation distance r = 10

RESULTS & DISCUSSIONS

Expression (12) representing, the propagation of strong spherical diverging shock through a non-uniform medium, shows its dependence on adiabatic index (γ), constant (ω) and propagation distance (r). It may be noted that for a given value of γ, increase or decrease in shock velocity dependence on ω. If $\omega > 4\gamma / \left\{2+\sqrt{2\gamma(\gamma-1)}\right\}$, shock velocity increases and If $\omega < 4\gamma / \left\{2+\sqrt{2\gamma(\gamma-1)}\right\}$, it decreases as shock progresses. For $\omega = 4\gamma / \left\{2+\sqrt{2\gamma(\gamma-1)}\right\}$ shock moves with constant strength.

Equation (13)-(15) represent the non-dimensional pressure, particle velocity and sound velocity immediately behind the shock of the medium as strong spherical diverging shock propagates, respectively.

Taking the initial shock strength $U/a_0 = 20$ at $r = 10$, for $\gamma = 1.33$ and $\omega = 2$, the variation of pressure, particle velocity, sound velocity, temperature and change in entropy with propagation distance (r), constant (ω) and adiabatic index (γ) are numerically calculated and displayed through figures (1-12).

It is found that all these parameters increase as shock propagates in the medium and also increase with increase in ω [fig. (1-6)]. Non dimensional shock velocity, pressure and particle velocity decrease as increase in adiabatic index (γ) [cf. Fig (7-9)] whereas sound velocity and temperature of the medium increase with increase in γ. [cf. Fig (10-11)].

The change in entropy with propagation distance (r) for $\omega = 1.9$, 2.0 and 2.1 has been shown in fig. (6). It is seen that change in entropy of medium increases as shock propagates. This agrees with earlier results [8, 13] as well as universal law of increase of entropy. The change in entropy increases with ω [cf. Fig. (6)] and decrease with γ [cf. Fig. (12)]. It is important to note that for $\omega = 4\gamma / \left\{2 + \sqrt{2\gamma(\gamma-1)}\right\}$, change in entropy as well as non-dimensional temperature is constant and this constant depends upon adiabatic index (γ) [cf. Fig. (16, 17)]. For values of $\omega < 4\gamma / \left\{2 + \sqrt{2\gamma(\gamma-1)}\right\}$, the change in entropy of medium decreases as shock diverges, is an important and peculiar feature.

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HOPF-BIFURCATION AND STABILITY OF A 2-PREDATORS, 1-PREY SYSTEM

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ABSTRACT

This paper considers a system modelling the interaction of two competing predatory populations with density-dependent death rats, living exclusively on a common prey. The existence of interior positive equilibrium (coexistence) is proven and sufficient conditions for its asymptotic stability are given. Furthermore, a derivation of an easily verifiable conditions for the existence of a hopf-bifurcating one-parameter family periodic solutions emanating from the interior positive equilibrium of a general of species predator-prey system are given. The stability of the bifurcating periodic 3-solutions is shown, through an example, to be an open question.

Key words: 3-species predator-prey system, hopf-bifurcation, equilibrium, periodic solution, stability.

Mathematics Subject Classification (2000) : 92 B05

INTRODUCTION

For a predator-prey system (see [1]) it is well known that the existence and the stability of a positive equilibrium is essential to investigate the existence of non-trivial periodic solutions. This question becomes even more interesting and important when two or more predatory populations compete for a single prey population. In this work, we have considered a system of three O.D.E.,s modelling the interaction of two predatory populations living exclusively on a common prey population. For both predatory populations, we have incorporated density-dependent death rates as opposed to constant death rates as considered in [3] and others, a situation that does not appear to be realistic. For a system with constant death rates, there may be no positive equilibrium existing and hence constant coexistence of the species is excluded. However, it is posibble that these competing predators together with their sole food source may coexist in a periodic fashion. In fact this does happen and the three species coexist in the form of state peiodicity see [10, 11].

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In the case, however of a system with density-dependent death rates, we have proved the existence of the interior positive equilibrium, its asymptotic stability, and existence of bifurcating periodic solutions emanating from this equilibrium point.

The organization of this paper is as follows.

In section 2, we give the model and discuss the asymptotic stability of equilibria in the co-ordinate planes.

Section 3 deals with the existence and asymptotic stability of the interior positive equilibrium. In section 4, we derive conditions for the existence of Hopf-bifurcating periodic orbits emanating from the interior positive equilibrium of a general 3-species system. We also show by an example that the question of stability of the bifurcating periodic orbit is inconclusive.

THE MODEL

Hypothesis on the Model

A system modelling the interspecific interactions of two competing predator populations and living exclusively on a common prey is of the form :

$$
\begin{aligned}
&\dot{x} = xf(x) - yp(x) - zq(x) \\
&\dot{y} = y[-g(y) + c_1 p(x)] \\
&\dot{z} = z[-h(z) + d_1 q(x)], \\
&x(0) = x_0 > 0,\ y(0) = y_0 > 0,\ z(0) = z_0 > 0.
\end{aligned}
\tag{2.1.1}
$$

where

$\left(\cdot = \frac{d}{dt}\right)$, c_1, d_1 are positive constants known as food (biomass) conversion rates.

We make the following assumptions, which are consistent with models of predator-prey system.

(H1) : $f(x)$; specific growth rate of prey x, $f(0) > 0$, $f'(x) \leq 0$ for $x \geq 0$, there is $k > 0$ (the environmental carrying capacity) such that $f(x)>0$ on $0 \leq x < k$, $f(k) = 0$, and $f(x) < 0$ on $x > k$.

(H2) : $p(x)$; the functional response of predator y with respect to prey x, and $p(0)=0$, $p'(x)>0$ for $x>0$.

(H3) : $q(x)$; the functional response of predator z with respect to prey x, and $q(0)=0$, $q'(x)>0$ for $x>0$.

(H4) : $q(x)=\alpha p(x)$, α is a positive constant.

(H5) : $g(y)$; density-dependent death rate of predator y, $g(0)>0$, and $g'(y)>0$ for $y>0$.

(H6) : $h(z)$; density-dependent death rate of predator z, $h(0)>0$, and $h'(z)>0$ for $z>0$.

Asymptotic Stability of Equilibria in Co-ordinate Planes

Clearly the system (2.1.1.) has an equilibrium at $E_0(0,0,0)$ which is hyperbolic and unstable along the x-axis.

By the assumption (H1), $E_1(k,0,0)$ [a predator-fee state] is also an equilibrium point. The eigenvalues of the variational matrix $V(E_1)$ of the system (2.1.1.) about $E_1(k,0,0)$ are :

$$\lambda = kf'(k)<0,\ \lambda_2=-g(0)+c_1p(k),\ \lambda_3=-h(0)+d_1q(k) \qquad (2.2.1)$$

Remark 2.2.1. From (2.2.1.) it follows that $E_1(k,0,0)$ is asymptotically stable along the x-axis. This implies that the prey population remains in a neighboured of k. For existence of $\hat{E}(\hat{x},\hat{y},0)$ and $\tilde{E}(\tilde{x},0,\tilde{z})$ as equilibria, it is necessary as well as sufficient that :

$$-g(0)+c_1p(k)>0 \text{ and } -h(0)+d_1q(k)>0 \qquad (2.2.2.)$$

as it implies increase of predator populations $y(t)$ and $z(t)$ respectively, see [4]. Thus we assume that each of the predators y and z can survive on the prey x in the absence of its rival, that is, there exist equilibria $\hat{E}(\hat{x},\hat{y},0)$ and $\tilde{E}(\tilde{x},0,\tilde{z})$ satisfying :

$$\left.\begin{aligned} \hat{x}f(\hat{x})-\hat{y}f(\hat{x})=0 \\ -g(\hat{y})+c_1p(\hat{x})=0 \end{aligned}\right\} \tag{2.2.3.}$$

and

$$\left.\begin{aligned} \tilde{x}f(\tilde{x})-\tilde{z}q(\tilde{x})=0 \\ -h(\tilde{z})+d_1q(\tilde{x})=0 \end{aligned}\right\} \tag{2.2.4.}$$

respectively.

Lemma 2.2.1. If $\frac{d}{dx}\left(\frac{xf(x)}{p(x)}\right)_{x=\hat{x}} < 0$ and $h(0)-d_1q(\hat{x})>0$, then the quilibrium $\hat{E}(\hat{x},\hat{y},0)$ is asymptotically stable.

Proof : The entries (a_{ij}) of the variational matrix $V(\hat{E})$ of the system (2.1.1.) about $\hat{E}(\hat{x},\hat{y},0)$ (where simplification is done using (2.2.3.)) are :

$$\left.\begin{aligned} &a_{11}=p(\hat{x})\frac{d}{dx}\left(\frac{xf(x)}{p(x)}\right)_{x=\hat{x}}<0, && a_{12}=-p(\hat{x})<0, && a_{13}=-q(\hat{x})<0 \\ &a_{21}=c_1\hat{y}p(\hat{x})>0, && a_{22}=-\hat{y}g'(\hat{y})<0, && a_{23}=0 \\ &a_{31}=0, && a_{32}=0, && a_{33}=-(h(0)-d_1q(\hat{x}))<0 \end{aligned}\right\} \tag{2.2.5.}$$

The characteristic equation of $V(\hat{E})$ is

$$\lambda^3+p_1\lambda^2+p_2\lambda+p_3=0, \tag{2.2.6.}$$

where

$$\left.\begin{aligned} p_1&=-(a_{11}+a_{22}+a_{23}) \\ p_2&=(a_{11}a_{22}+a_{22}a_{33}+a_{11}a_{33})-(a_{12}a_{21}+a_{13}a_{31}+a_{23}a_{32}) \\ p_3&=(a_{11}a_{23}a_{32}+a_{22}a_{31}a_{13}+a_{33}a_{12}a_{21})-(a_{11}a_{22}a_{23}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}) \end{aligned}\right\}. \tag{2.2.7.}$$

From (2.2.5.) and (2.2.7.), we have

$$p_1 > 0,\ p_3 > 0,\ \text{and}\, p_1 p_2 - p_3 > 0.$$

Thus by Routh-Hurwitz criterion, $\hat{E}(\hat{x},\hat{y},0)$ is asymptotically stable.

Lemma 2.2.2. If $\frac{d}{dx}\left(\frac{xf(x)}{p(x)}\right)_{x=\tilde{x}} < 0$ and $g(0) - c_1 p(\tilde{x}) > 0$, then the equilibrium $\tilde{E}(\tilde{x},0,\tilde{z})$ is asymptotically stable.

Proof : The proof is similar to that of lemma 2.2.1.

Remark 2.2.2. There is no equilibrium in the y-z plane, for if we suppose that there does exist an equilibrium $E^*(0,y^*,z^*)$ in the y-z plane which is given by $g(y^*) = 0$ and $h(z^*) = 0$, then this contradicts assumptions (H5) and (H6). Moreover, it is, indeed, consistent with the intuitive ecological observation that predators can not coexist while their sole food source is extinct.

EXISTENCE OF INTERIOR POSITIVE EQUILIBRIUM AND ITS STABILITY

For the system (2.2.1.) with $g(y)$ and $h(z)$ as contant functions, the question whether or not it is possible to have all three species x, y and z coexisting is a quite significant and interesting question both from mathematical and biological points of view.

In general, no steady state $x \neq 0,\ y \neq 0,\ z \neq 0$ exists in this case.

This is because the algebraic equations (say with Holling type II functional responses)

$$xf(x) - yp(x) - zq(x) = 0$$

$$-s + c_1 p(x) = 0$$

$$-d + d_1 q(x) = 0$$

do not have a simultaneous solution. In the degenerate case, $x_1 = x_2 < k$, there exists a line segment of steady states joining $\hat{E}(\hat{x},\hat{y},0)$ and $\tilde{E}(\tilde{x},0,\tilde{z})$. This implies, in general, that the competing predators would exclude coexistence in a constant steady state situation. Since no constant equilibrium exists, this could occur only through a stable time varying solutions. In fact, this does happen and the ghree species can coexist in the form of a stable periodic trajectory see [10, 11].

However, when both $g(y)$ and $h(z)$ are not constant functions, there does exist an interior positive equilibrium as we show now.

Existence of Positive Equilibrium

For the existence of positive equilibrium, we give an indirect proof. If such a positive equilibrium $E^*(x^*,y^*,z^*)$ of the system (2.1.1.) exists then it is given by :

$$\begin{aligned} &x^*f(x^*)-y^*p(x^*)-z^*q(x^*)=0 \\ &-g(y^*)+c_1p(x^*)=0 \qquad (3.1.1.) \\ &-h(z^*)+d_1q(x^*)=0 \end{aligned}$$

As shown in [2, 4] the notions of global asymptotic stability and persistence of the system are connected in the sense that the criteria for persistence of the system (i.e. the persistence of top predator populations in a simple food chain. Indeed, in our system (2.1.1.) each predator acts as a top predator, consequently imply the existence of a positive equilibrium.

Thus, for the existence of positive equilibrium we should have the following conditions:

(1) Range $(g)=c_1$ Range (p), Range $(h)=d_1$ Range (g), and g^{-1} and h^{-1} both exist.

(2) $\frac{xf(x)}{p(x)}$ and $\frac{xf(x)}{q(x)}$ are both strictly decreasing functions in the positive cone.

This for the equilibrium to be globally asymptotically stable (see Remark 3.2.1.).

(3) $-g(0)+c_1p(\tilde{x})>0$ and $-h(0)+d_1q(\hat{x})>0$ where $\hat{E}(\hat{x},\hat{y},0)$ and $\tilde{E}(\tilde{x},0,\tilde{z})$ are equilibria in the co-ordinate planes (see section 2.2.). These two conditions are for persistence of the system (2.1.1) (see [2,3,4]).

Remark 3.1.1.

The condition that $\frac{xf(x)}{p(x)}\left(\frac{xf(x)}{q(x)}=\frac{xf(x)}{\alpha p(x)} \text{ when (H4) holds}\right)$ is strictly decreasing also implies global asymptotic stability of the equilibria $\hat{E}(\hat{x},\hat{y},0)$ and $\tilde{E}(\tilde{x},0,\tilde{z})$, that is non-existence of limit cycles in the x-y plane and x-z plane.

Thus conditions (1), (2), (3) imply the existence of positive equilibrium.

We illustrate this by two examples.

Example 3.1.1.

Consider the system (2.1.1.) with logistic growth of the prey population and Holling type II functional responses.

$$\dot{x}=x(1-x)-y\frac{2x}{1+x}-z\frac{4x}{1+x}$$

$$\dot{y}=y\left[-(1+y)+\frac{33}{16}\frac{2x}{1+x}\right]. \qquad (3.1.2.)$$

$$\dot{z}=z\left[-(1+z)+\frac{11}{9}\frac{4x}{1+x}\right]$$

Condition (1) : is trivially satisfied.

Condition (2) :

$$\frac{d}{dx}\left(\frac{xf(x)}{p(x)}\right)=-x<0, \qquad 0<x^{*}\le x<k$$

$$\frac{d}{dx}\left(\frac{xf(x)}{q(x)}\right)=-\frac{1}{2}<0, \qquad 0<x^{*}\le x<k$$

for condition (3) :

$$\hat{E}(\hat{x},\hat{y},0)=\hat{E}\left(\frac{1}{2},\frac{3}{8},0\right)$$

$$\tilde{E}(\tilde{x},0,\tilde{z})=\tilde{E}\left(\frac{1}{3},0,\frac{8}{36}\right)$$

$$-g(0)+c_1p(\tilde{x})=-1+\frac{33}{16}\left(p\left(\frac{1}{3}\right)\right)=\frac{1}{32}>0$$

$$-h(0)+d_1q(\hat{x})=-1+\frac{11}{9}q\left(\frac{1}{2}\right)=\frac{17}{27}>0,$$

and we have $E^*(x^*,y^*,z^*)=E^*(.329,.02,.21)$ a positive equilibrium for (3.12.).

Example 3.1.2.

Consider the system :

$$\begin{aligned}\dot{x}&=x(1-x)-yx-2xz\\ \dot{y}&=y[-2(2+y)+3x]\\ \dot{z}&=z[-(2+z)+4x]\end{aligned}\tag{3.1.3.}$$

condition (2) is satisfied.

$$\hat{E}(\hat{x},y,0)=\hat{E}\left(\frac{3}{4},\frac{1}{4},0\right)$$

$$\tilde{E}(\tilde{x},0,\tilde{z})=\hat{E}\left(\frac{5}{9},0,\frac{5}{9}\right)$$

$-h(0)+d_1q(\hat{x})=1>0$, but

$-g(0)+c_1p(\tilde{x})=-2+3.\frac{5}{9}=-\frac{1}{3}<0,$ thus (3) is not satisfied implying that (3.1.3.) may have no positive equilibrium.

No solving

$$\left.\begin{aligned}1-x-y-2z&=0\\ 2+y&=3x\\ 2+z&=4x\end{aligned}\right\}\Rightarrow x=\frac{7}{12},y=-\frac{1}{4},z=\frac{1}{3}$$

which in not a positive equilibrium of (3.1.3.).

3.2. Asymptotic Stability of $E^*(x^*,y^*,z^*)$.

The asymptotic stability of the equilibrium $E^*(x^*,y^*,z^*)$ for (2.1.1.) is proved under the assumption (H4), that is

$$q(x)=\alpha p(x). \tag{3.2.1.}$$

where α is taken as a positive parameter which will be of significance for the Hopf-bifurcation in section 4.

Theorem 3.2.1.

If $\frac{d}{dx}\left(\frac{xf(x)}{p(x)}\right)_{x=x^*}<0,$ then the equilibrium $E^*(x^*,y^*,z^*)$ is asymptotically stable.

Proof :

The entries (a_{ij}) of the variational matrix $V(E^*)$ of the system (2.1.1.) about $E^*(x^*,y^*,z^*)$ (as simplification is done using (3.1.1.)) are

$$\left.\begin{aligned} &a_{11} = p(x^*)\frac{d}{dx}\left(\frac{xf(x)}{p(x)}\right)_{x=x^*} < 0, && a_{12} = -p(x^*) < 0, && a_{13} = -\alpha p(x^*) < 0 \\ &a_{21} = y^* c_1 p(x^*) > 0, && a_{22} = -y^* g'(y^*) < 0, && a_{23} = 0 \\ &a_{31} = \alpha d_1 z^* p(x^*) > 0, && a_{32} = 0, && a_{33} = -z^* h'(z^*) < 0 \end{aligned}\right\} \quad (3.2.2.)$$

From (2.2.6.) and (3.2.2.) we have $p_1 > 0, p_3 > 0$ and $p_1 p_2 - p_3 > 0$.

Thus by the Routh-Hurwitz criterion, $E^*(x^*, y^*, z^*)$ is asymptotically stable.

Remark 3.2.1.

For the global asymptotic stability of $E^*(x^*, y^*, z^*)$, we define a Lyapunov function on $\{(x,y,z):0 < x^* \le x < k, 0 < y^* \le y < \beta, 0 < z^* \le z < \gamma, \beta > 0, \gamma > 0\}$ as

$$V(x,y,z) = \int_{x^*}^{x} (c_1 + d_1)\left(1 - \frac{p(x^*)}{p(w)}\right) dw + \int_{y^*}^{y} \left(\frac{w - y^*}{w}\right) dw + \int_{z^*}^{z} \left(\frac{w - z^*}{w}\right) dw \quad (3.2.3.)$$

It is straightforward to show that

$\dot{V}(x,y,z) < 0$ provided $\dfrac{xf(x)}{p(x)}$ is a strictly decreasing function on $0 < x^* \le x < k$.

HOPF-BIFURCATION IN 3-SPECIES SYSTEM

Hopf-bifurcation is one of the most common ways through which a constant steady state loses its stability giving rise to a solution (or a family of solutions) of a periodic nature.

The technique of Hopf-bifurcation relies heavily on the effects induced by the parameter incorporated in the system on the stability of the steady state of the system. Thus, Hopf-bifurcation is characterized by a stability change of the steady state accompanied by the creation of limit cycles.

The Hopf-bifurcation theorem is a very powerful general tool in establishing the existence of a one-parameter family of periodic solutions and in a neighborhood of the bifurcation point, it also gives their periods. However, it does not immediately provides any information about the stability of these bifurcating periodic solutions.

Various successful attempts have been made to ascertain the orbital stability of the bifurcating periodic solution such as stability formula as in [6], the procedure applied in [5], center manifold theory and the constant "a" value technique in [12], and the method of Negrini-Salvadori [7]. Nevertheless, we show by an example that the stability of the bifurcating periodic solution seems to be an open question.

Conditions for Existence of Bifurcating Periodic Solutions

In this section we obtain conditions in terms of p_1, p_2, p_3 as defined in (2.2.7.) for the existence of bifurcating periodic solution emanating from the positive equilibrium $E^*(x^*, y^*, z^*)$ of the general 3-species system

$$\dot{x}_i = x_i f_i(x_i, y_i, z_i, \alpha), \quad i = 1,2,3 \tag{4.1.1.}$$

α being a parameter.

We shall be seeking conditions on the characteristic equation of the variational matrix $V(E^*)$, i.e.

$$\lambda^3 + p_1\lambda^2 + p_2\lambda + p_3 = 0 \tag{4.1.2.}$$

to have 2 purely imaginary roots and a criterion so as the transversality condition holds.

Suppose (4.1.2.) has one real root λ_1 and two complex conjugate roots :

$$\lambda_2 = \lambda_r + i\lambda_s, \quad \lambda_3 = \lambda_r - i\lambda_s.$$

Thus

$$\lambda^3 + p_1\lambda^2 + p_2\lambda + p_3 = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)$$

$$\lambda^3 - \lambda^2(2\lambda_r + \lambda_1) + \lambda\left(|\lambda_2|^2 + 2\lambda_r\lambda_1\right) - |\lambda_2|^2\lambda_1.$$

By comparing coefficients of like terms, we have

$$\left.\begin{aligned} -p_1 &= 2\lambda_r + \lambda_1 \\ p_2 &= |\lambda_2|^2 + 2\lambda_r\lambda_1 \\ p_3 &= -|\lambda_2|^2\lambda_1 \end{aligned}\right\} \qquad (4.1.3.)$$

Let us consider α, as given in (3.2.1.), to be the Hopf-bifurcation parameter which induces (by tunning it about a critical value) the change in stability status of the equilibrium $E^*(x^*, y^*, z^*)$.

Now (4.1.2.) has to have two purely imaginary roots if $\lambda_r = 0$.

Thus from (4.1.3.) we have

$$p_1, p_2 - p_3 = 0 \qquad (4.1.4.)$$

Equation (4.1.4.) gives the critical value of α, call it α_c, such that $p_1 p_2 - p_3 = 0$.

Thus α_c satisfies $\bar{p}_1, \bar{p}_2 - \bar{p}_3 = 0$,

where $\bar{p}_i (i = 1,2,3)$ denotes the value of p when α is replaced by α_c.

Thus the two purely imaginary roots are $\pm\, iw$, where $w = \sqrt{p_2}$ whit $p_2 > 0$. This is because if λ_1 is a real root of (4.1.2.) then the other two are give by

$$\lambda_{2,3} = \left[-(\lambda_1 + p_1) \pm \sqrt{(\lambda_1 + p_1)^2 - 4(\lambda_1^2 + \lambda_1 p_1 + p_2)}\right] \Big/ 2$$

If $\lambda_{2,3}$ are purely imaginary then $\lambda_1 = -p_1$ and it implies that $\lambda_{2,3} = \pm i\sqrt{p_2}$ with $p_2 > 0$.

Next to find a transversality condition in terms of p_1, p_2, p_3 such that $\frac{d}{dx}(\lambda_r(\alpha))_{\alpha=\alpha_c} \neq 0$

Now eliminating λ_1 from (4.1.3.) will result in

$$8\lambda_r^3 + 8\lambda_r^2 p_1 + 2\lambda_r\left[p_1^2 + p_2\right] + p_1 p_2 - p_3 = 0 \qquad (4.1.5.)$$

Differentiating (4.1.5.) with respect to α, then putting $\alpha = \alpha_c$, and observing that $\lambda_r(\alpha_c) = 0$, $(p_1p_2 - p_3)_{\alpha=\alpha_c} = 0$, we get

$$2\left[p_1^2 + p_2\right]_{\alpha=\alpha_c} \frac{d}{d\alpha}(\lambda_1(\alpha))_{\alpha=\alpha_c} = -\frac{d}{d\alpha}(p_1p_2 - p_3)_{\alpha=\alpha_c}$$

$$\therefore \frac{d}{d\alpha}(\lambda_r(\alpha))_{\alpha=\alpha_c} = -\frac{1}{2\left[p_1^2 + p_2\right]}\frac{d}{d\alpha}(p_1p_2 - p_3)_{\alpha=\alpha_c} \tag{4.1.6.}$$

Hence

$$p_1^2 + p_2 > 0, \ \frac{d}{d\alpha}(\lambda_r(\alpha)) > 0(<0) \text{ if } \frac{d}{d\alpha}(p_1p_2 - p_3) < 0(<0) \tag{4.1.7.}$$

Remark 4.1.1.

Since for $d = \frac{d}{d\alpha}(\lambda_r(\alpha))_{\alpha=\alpha_c} > 0$, the eigenvalues cross from left half-plane to the right-half plane as α increases, it follows that the equilibrium is asymptotically stable for $\alpha < \alpha_c$ and unstable for $\alpha > \alpha_c$.

For $d < 0$, the eigenvalues cross from the right half-plane to the left half-plane as α increases, it follows that the equilibrium point is unstable for $\alpha < \alpha_c$ and asymptotically stable for $\alpha > \alpha_c$.

This means that for $d > 0 \left(i.e. \ \frac{d}{d\alpha}(p_1p_2 - p_3) < 0 \right)$, if $F(\alpha) = p_1p_2 - p_3$, then $F(\alpha)$ is decreasing at $\alpha = \alpha_c$, which implies that $p_1p_2 - p_3 < 0$ for $\alpha > \alpha_c$, $\alpha - \alpha_c$ to be small.

Thus with $p_1 > 0$ and $p_3 > 0$, the asymptotic stability of $E^*(x^*, y^*, z^*)$ depends upon $p_1p_2 - p_3$. Hence, for $\alpha < \alpha_c, p_1p_2 - p_3 > 0$ we have $E^*(x^*, y^*, z^*)$ asymptotically stable, while for $\alpha > \alpha_c$, $\alpha - \alpha_c$ is small, $p_1p_2 - p_3 < 0$, we have $E^*(x^*, y^*, z^*)$

unstable, whereas for $\alpha = \alpha_c$, $p_1 p_2 - p_3 = 0$ there is a Hopf-bifurcation.

Thus we have proved the following theorem :

Theorem 4.1.1.

The interior equilibrium point in the positive cone of the system (2.1.1.) is asymptotically stable for $\alpha < \alpha_c$, unstable for $\alpha > \alpha_c$, and there is a Hopf-bifurcation at $\alpha = \alpha_c$, where α_c is given by (4.1.4.).

However; for $\frac{d}{d\alpha}(p_1 p_2 - p_3)_{\alpha=\alpha_c} > 0\left(i.e. \frac{d}{d\alpha}(\lambda_r(\alpha))_{\alpha=\alpha_c} < 0\right)$, then theorem 4.1.1. reads as :

Theorem 4.1.2.

The interior equilibrium point in the positive cone of the system (2.1.1.) is asymptotically stable for $\alpha > \alpha_c$, unstable for $\alpha < \alpha_c$, and there is a Hopf-bifurcation at $\alpha = \alpha_c$, where α_c is given by (4.1.4.)

We illustrate the two theorem by the following examples.

Example 4.1.1.

Consider the sysem (The classical Lorenz equations), see [6]

$$\begin{aligned} \dot{x} &= -\sigma x + \sigma y \\ \dot{y} &= -xz + ry - y \qquad (4.1.8.) \\ \dot{z} &= xy + -bz \end{aligned}$$

where b and σ are positive constants and r to be taken as a Hopf-bifurcation parameter. $E^*(x^*, y^*, z^*) = E^*\left(\sqrt{b(r-1)}, \sqrt{b(r-1)}, (r-1)\right)$, whit $r > 1$.

Moreover, $p_1 = 1 + b + \sigma$, $p_2 = (r + \sigma)b$, $p_3 = 2\sigma b(r-1)$.

$p_1 p_2 - p_3 = 0$ implies that the critical value of the parameter r is

$$r_c = \frac{\sigma(\sigma+b+3)}{\sigma-(b+1)},\ \frac{d}{dr}(p_1p_2-p_3)_{r=r_c} = b(1+b-\sigma)<0 \text{ provided that } \sigma>1+b.$$

Thus Hopf-bifurcation will occur at $r=r_c$ provided that $\sigma>1+b$ and this condition is the same as in [6].

Example 4.1.2.

Consider the following predator-prey system, see [8] :

$$\dot{x} = x\left(1-\frac{\alpha k}{\in}y-z\right)$$

$$\dot{y} = \frac{y}{\in}(-y+\beta kz) \tag{4.1.9.}$$

$$\dot{z} = \frac{a}{\in}(x-z)$$

which is a predator-prey model with time lag, where $\alpha, k, \in, \gamma, \beta$ are positive constants and $a>0$ be taken as the Hopf-bifurcation parameter. The positive equilibrium point is $E^*(x^*,y^*,z^*)=E^*\left(\frac{\gamma}{\beta k},\frac{\in}{\alpha k}\left(1-\frac{\gamma}{\beta k}\right),\frac{\gamma}{\beta k}\right)$ with $\beta k-\gamma>0$.

Thus we get

$$p_1=\frac{a}{\in},\ p_2=\frac{a\gamma}{\in\beta k},\ p_3=\frac{\gamma a}{\in^2\beta k}(\beta k-\gamma), \text{ hence}$$

$$p_1p_2-p_3=\frac{a\gamma}{\in^2\beta k}(a-\beta k+\gamma)=0 \text{ implies } a_c=\beta k-\gamma>0$$

(a_c is the parameter critical value at which Hopf-bifurcation occurs).

For the transversality condition we have :

$\frac{d}{da}(p_1p_2 - p_3)_{a=a_c} = \frac{\gamma}{\in^2 \beta k}(\beta k - \gamma) > 0$ which implies $\frac{d}{d\alpha}(\lambda_r(a))_{a=a_c} < 0$ and hence the equilibrium point $E^*(x^*, y^*, z^*)$ is asymptotically stable for $a > a_c$, unstable for $a < a_c$ and Hopf-bifurcation occurs at $a = a_c$.

Stability of Bifurcating Periodic Orbit

In this section we illustrate by an example that the stability of the bifurcating periodic solution seems to be an open question.

Example 4.2.1.

Consider a 3-species predator-prey model in which two species (x and y say) at the bottom are feeding on each other and at the same time are fed upon by the third species (z) acting as the top predator of the system. This system can be represented as :

$$\left.\begin{aligned} \dot{x} &= x[1 - x + \alpha y - \beta z] \\ \dot{y} &= y[1 - y + \alpha x - \beta z] \\ \dot{z} &= z[-1 - z + \alpha x + \alpha y] \end{aligned}\right\} \qquad (4.2.1.)$$

where α, β positive constants (α represents the biomass conversion rate in all cases), α is taken as the Hopf-bifurcation parameter.

The interior positive equilibrium $E^*(x^*, y^*, z^*)$ is given by

$$x^* = y^* = \frac{1+\beta}{1-\alpha+2\alpha\beta},\ z^* = \frac{3\alpha-1}{1-\alpha+2\alpha\beta} \text{ with } \alpha > 1. \qquad (4.2.2.)$$

Computing p_1, p_2, p_3, $p_1p_2 - p_3$, and then from $p_1p_2 - p_3 = 0$, we get

$$\alpha_c = \frac{\beta}{\beta - 2} \text{ with } \beta > 2. \qquad (4.2.3.)$$

Solving the characteristic equation

$\lambda^3 + p_1\lambda^2 + p_2\lambda + p_3 = 0$, and $\alpha = \alpha_c$ we get the eigenvalues as

$$\lambda_1 = -1,\ \lambda_{2,3} = \pm i w_0,\ \text{where } w_0 = \sqrt{\frac{\beta+1}{\beta-1}}. \qquad (4.2.4.)$$

$$\frac{d}{d\alpha}(p_1p_2 - p_3)_{\alpha=\alpha_c} = \frac{-(\beta-2)^2(\beta+1)\beta}{(\beta^2-1)^2} < 0, \text{ which implies,}$$

$$\frac{d}{d\alpha}(\lambda_1(\alpha))_{\alpha=\alpha_c} = \frac{1}{4}\frac{(\beta-2)^2}{(\beta^2-1)} > 0 \quad .$$

See figure 4.2.1.

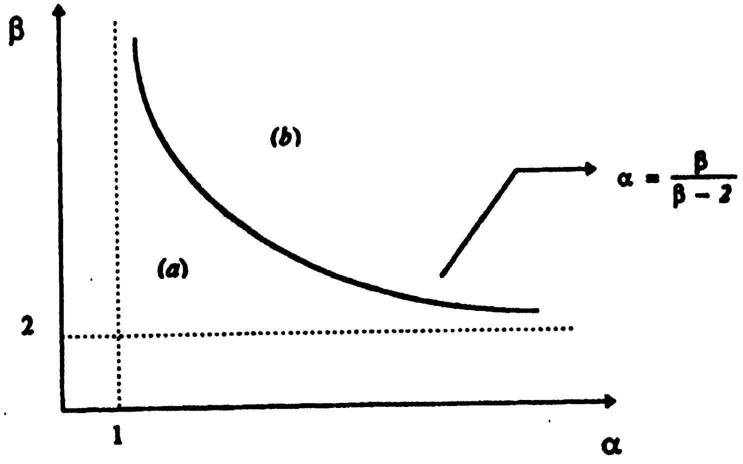

Figure 4.2.1.

The figure shows the stability status of (4.2.1.) with respect to (E^*) *as a function of the interaction parameters* α and β. *In the region (a), the positive equilibrium point (at which all the three species may coexist) is asymptotically stable. Hopf-bifurcating periodic solutions are obtained for all* α and β *lying on the curve in the region (b), the equilibrium is unstable.*

(i) The procedure as in [5] and [9]

Next we move the origin into the equilibrium $E^*(x^*, y^*, z^*)$ by co-ordinate transformation and performing the parameter transformation (4.2.3.) at the same time, we get the following system which is equivalent to the system (4.2.1.) :

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \frac{1}{2(\beta-1)} \begin{pmatrix} -(\beta-2) & \beta & -\beta(\beta-2) \\ \beta & -(\beta-2) & -\beta(\beta-2) \\ \dfrac{2\beta}{\beta-2} & \dfrac{2\beta}{\beta-2} & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + diag(x,y,z) \begin{pmatrix} -1 & \dfrac{\beta}{\beta-2} & -\beta \\ \dfrac{\beta}{\beta-2} & -1 & -\beta \\ \dfrac{\beta}{\beta-2} & \dfrac{\beta}{\beta-2} & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} . \tag{4.2.5.}$$

The eigenvectors corresponding to the eigenvalues $\lambda_j(\alpha_c)\, j = 1,2,3$ are

$S_1 = col[1,-1,0]$, $S_{2,3} = col[\beta-2,\ \beta-2,\ 2\beta] \pm i\, col[w_0(\beta-1)(\beta-2), w_0(\beta-1)(\beta-2), 0]$.
Introducing S_1 together with real and imaginary parts of S_3 as a new base, we perform the co-ordinate transformation

$$col[x,y,z] = \begin{pmatrix} \beta-2 & -w_0(\beta-1)(\beta-2) & 1 \\ \beta-2 & -w_0(\beta-1)(\beta-2) & -1 \\ 2\beta & 0 & 0 \end{pmatrix} col[u,v,w].$$

In the new co-ordinate system (4.2.5.) has the form :

$$\begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} 0 & -w_0 & 0 \\ w_0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} + \begin{pmatrix} F^1 \\ F^2 \\ F^3 \end{pmatrix} \tag{4.2.6.}$$

where

$$\left.\begin{aligned} F^1 &= -2w_0\beta uv \\ F^2 &= w_0(\beta-2)u^2 - \left(\frac{\beta^2(\beta-2)+4}{\beta-1}\right)uv - w_0(\beta-2)v^2 + \frac{w^2}{w_0(\beta-1)(\beta-2)} \\ F^3 &= -(\beta-2)(\beta+2)uw + w_0(\beta-2)^2 vw \end{aligned}\right\} \tag{4.2.7.}$$

The procedure given in [5] and in [9] is as follows;

The sign of the constant $C_1(\alpha_c)$ defined in terms of partial derivatives of (4.2.7.) determines the stability and direction of the bifurcation of the periodic solution. On calculation of $C_1(\alpha_c)$ (see Appendix), it is found that $\mathrm{Re}(C_1(\alpha_c)) = 0$, and hence no information concerning orbital stability of the bifurcating periodic solution.

(ii) Reduction of the System to the Center Manifold

We apply (as an alternative method) center manifold theorem which simply says that the orbital structure near $E^*(x^*, y^*, z^*)$ with α replaced by α_c is determined by the vector field (4.2.1.) restricted to the center manifold.

The center manifold can locally be represented as

$$\{(u, v, w) \in R^3 : w = h(u,v) \in h(0,0) = 0, h_u(0,0) = h_v(0,0)\} \tag{4.2.8.}$$

we choose $h(u,v)$ as

$$h(u,v) = \frac{1}{2}\left[h_{11}u^2 + 2h_{12}uv + h_{22}v^2\right] + o(3).$$

If $(u(s), v(s), w(s))$ is a solution of (4.2.6.) near the origin with a value on the center manifold, that is

$$w(s) = h(u(s), v(s)), \text{ then}$$

$$\dot{w}(s) - h_u(u,v)\dot{u} - h_v(u,v)\dot{v} \equiv 0 \tag{4.2.9.}$$

Using (4.2.6.) in (4.2.9.), omitting terms of order at least three and equating coefticients to zero, we get

$h_{11} = h_{12} = h_{22} = 0$. This implies $w = h(u,v) = 0$.

Thus (4.2.6.) on center manifold becomes

$$\left.\begin{aligned}\dot{u} &= -w_0 v - 2w_0\beta uv\\ \dot{v} &= w_0 u + w_0(\beta-2)u^2 - \left(\frac{\beta^2(\beta-2)+4}{\beta-1}\right)uv - w_0(\beta-2)v^2\end{aligned}\right\} \qquad (4.2.10.)$$

On calculating the constant "*a*" see [12] or the Index "*I*" (see Appendix), it is found that "*a*" (or *I*) = 0, and hence no information regarding orbital stability of the bifurcating periodic solution.

(iii) The Negrini-Salvadori Method.

As our third attempt to study orbital stability of the bifurcating periodic solution, we apply the method of Negrini-Salvadori [7] in which a Lyapunov function is assumed in the form :

$$V(u,v) = u^2 + v^2 + A(u,v) + B(u,v),$$

where *A* and *B* are homogeneous polynomials of order three and order four respectively.

$$A(u,v) = a_1u^3 + a_2u^2v + a_3uv^2 + a_4v^3$$

$$B(u,v) = b_1u^4 + b_2u^3v + b_3u^2v^2 + b_4uv^4 + b_5v^4.$$

The coefficient a_i and b_j are to be determined in such a way that the derivative of *V* with respect to the system (4.2.10.) is

$$\dot{V}(u,v) = G(u^2+v^2)^2 + o(5) \qquad (4.2.11.)$$

where the constant *G* is uniquely determined and the orbital stability of the bifurcating periodic solution depends on the sign of *G*.

It turns out (again) that $G = 0$, and hence (again) no information regarding orbital stability of the bifurcating periodic solution of (4.2.1.) can be deduced.

APPENDIX

(I) Calculation of $C_1(\alpha_c)$ based on Kazarinoff recipe-summary (see [5,9] Using (4.2.7.), i.e.

$$F^1 = -2w_0\beta xy$$

$$F^2 = w_0(\beta-2)x^2 - \left[\frac{\beta^2(\beta-2)+4}{\beta-1}\right]xy - w_0(\beta-2)y^2 + \frac{z^2}{w_0(\beta-1)(\beta-2)}$$

$$F^3 = -(\beta-2)(\beta+2)xz + w_0(\beta-2)^2 yz$$

$$g_{11} = \left(F^1_{xx} + F^1_{yy}\right)/4 + i\left(F^2_{xx} + F^2_{yy}\right)/4 = 0$$

$$g_{02} = \left(F^1_{xx} - F^1_{yy} - 2F^2_{xy}\right)/4 + i\left(F^2_{xx} - F^2_{yy} + 2F^1_{xy}\right)/4 = \frac{\beta^2(\beta-2)+4}{2(\beta-1)} - 2iw_0$$

$$g_{20} = \left(F^1_{xx} - F^1_{yy} + 2F^2_{xy}\right)/4 + i\left(F^2_{xx} - F^2_{yy} + 2F^1_{xy}\right)/4 = \frac{\beta^2(\beta-2)+4}{2(\beta-1)} + 2iw_0(\beta-1)$$

$$g_{21} = \left[F^1_{xz} + F^2_{yz} + i\left(F^2_{xz} - F^1_{yz}\right)\right]\left[F^3_{xx} + F^3_{yy}\right]/4 + 2^{-1}\left[F^1_{xz} - F^2_{yz} + i\left(F^1_{yz} + F^2_{xz}\right)\right]$$

$$\left[F^3_{xx} - F^3_{yy} - 2iF^3_{xy}\right]/\left(4 + 2iw_0\right) = 0$$

$$C_1(\alpha_c) = i\left(g_{20}g_{11} - 2|g_{11}|^2 - 3^{-1}|g_{02}|^2\right)/2w_0 + g_{21}/2 =$$

$$i\left[-\frac{1}{3}\left(\frac{\beta^2(\beta-2)+4}{2(\beta-1)}\right)^2 + 4w_0^2\right]$$

Hence $\operatorname{Re} C_1(\alpha_c) = 0$

(II) Calculation of the constant "a"
(see [12, page 277])

$$a = \frac{1}{16}\left[f^1_{xxx} + f^1_{xyy} + f^2_{xxy} + f^2_{yyy}\right] + \frac{1}{16w_0}$$

$$\left[f^1_{xy}\left(f^1_{xx} + f^1_{yy}\right) - f^2_{xy}\left(f^2_{xx} + f^2_{yy}\right) - f^1_{xx}f^2_{xx} + f^1_{yy}f^2_{yy}\right]$$

where all partial derivatives are evaluated at the bifurcation point.

Now $f^1 = -2w_0\beta xy, f^2 = w_0(\beta-2)x^2 - \left[\frac{\beta^2(\beta-2)+4}{\beta-1}\right]xy - w_0(\beta-2)y^2.$

Thus $a = 0$.

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PERSISTENCE IN A 3-SPECIES PREDATOR-PREY MODEL

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ABSTRACT

The ecological notion of system persistence in modeling the interaction of two competing predatory populations living exclusively on a common prey, is investigated.

Freedom and Waltman [4], and El-Owaidy and Ammar [2] have discussed the persistence of such models based upon the assumption of nonexistence of limit cycles. In this paper, however; the nonexistence of limit cycles is proven, first, by global asymptotic stability of equilibria through the construction of a suitable Lyapunov function, and second, persistence criteria of the system are obtained.

Key words: Predator-prey system, Lyapunov function, limit cycle, asymptotic global stability.

Mathematics Subject Classification (2000) : 92 B05

INTRODUCTION

One of the most interesting questions in ecological models concerns the survival of all species of the model. This question of persistence becomes more important when two or more species compete for a single prey species.

The significant concepts of permanence and persistence, both exclude extinction of species for all positive initial conditions. In biological terms, persistence means that the density of each population remains, asymptotically, above a positive bound independent of the initial conditions, i.e.all species stay away from extinction.

Mathematically, this may be stated in terms of behaviour of solutions of the model which represents the biological phenomenon.

An ecological differential system

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$\dot{x}_i = x_i f(x_1, x_2, \ldots x_n) \quad \text{for } i = 1,2,\ldots,n$

is said to be permanent or uniformly persistent if there exists a compact set k in the interior of $R^n_+ = \{x \in R^n : x_i \geq 0$ for $1 \leq i \leq n$ such that all orbits end up in k. This guarantees that the number of each population $x_i(t)$ is bounded away from zero if $x_i(0) >$ for all i.

By the term weak persistence, we mean that for all i, $\limsup_{t \to +\infty} x_i(t) > 0$ whenever $x(0)>0$. Under this "*lim sup*" definition of persistence, a population can frequently approach extinction. By the term strong persistence, we mean that for all i, $\liminf_{t \to +\infty} x_i(t) > 0$ whenever $x(0)>0$.

We have applied the latter definition of persistence as used in Freedman and Waltman [4]. This difinition of persistence was reformulated in [4] as follows :

"A persistence with initial conditions in the positive cone will persist if there are no $\omega -$ *limit* points of the solution on the boundary of the positive cone".

This means if □ (X) be the orbit through the point $X = (x, y, z)$ with $x>0, y>0, z>0$, and if $\Omega(X)$ be the $\omega -$ *limit* set of □ (X), then $\Omega(X)$ be interior to the positive cone.

Since, the question of two predator populations competing for a single prey population has occupied an important place in ecological literature, so in this paper, we have considered a system modeling the interactions of two competing predator populations living exclusively on a common prey. For both the predator populations we have taken density-dependent death rates and in this sense, we have extended the result of El-Owaidy and Ammar [2].

Freedman and Waltman [4] in theorem 4.1., obtained persistence criteria by an assumption that for such a system, there are no limit cycles surrounding an interior equilibrium in a co-ordinate plane.

We have actually proved the non-existence of limit cycles by constructing Lyapunov functions for the equilibria and have obtained conditions under which the equilibria are are globally asymptotically stable. This implies that all the trajectories of the system in the positive cone will spiral toward this equilibrium point, that is non-existence of limit cycles.

Then, we have applied the persistence criteria as given in theorem 4.1. under reference, to obtain conditions for persistence of our system. Thus our result of persistence seems to be more comprehensive than that in theorem 4.1. of Freedman and Waltman [4].

The organization of this paper is as follows :

In section 2, we give the model. Section 3 deals with the global asymptotic stability of the problem. Section 4 deals with persistence results and we illustrate these by an example.

THE MODEL AND EQUILIBRIA

A system modeling the interaction of two competing predator populations *y* (*t*) and *z* (*t*) living exclusively on a common prey *x* (*t*).

$$\dot{x} = xf(x) - yp(x) - zq(x)$$

$$\dot{y} = y[-g(y) + c_1 p(x)] \qquad (2.1)$$

$$z = z[-h(z) + d_1 q(x)],$$

$$x(0) = x_0 > 0, y(0) y_0 \geq 0, z(0) = z_0 \geq 0.$$

Where $\left(\cdot = \frac{d}{dt} \right)$, c_1, d_1 are positive constants, known as food conversion rates.

We make the following assumptions, which are consistent with models of predator-prey systems.

(H1) : $f(x)$; specific growth rate of prey x

$f(0) > 0, f'(x) \leq 0$ for all $x > 0$, there exists $k > 0$ (the carrying capacity) such that $f(x) > 0$ on $0 \leq x < k, f(k) = 0$ and $f(\mathrm{x}) < 0$ on $x > k$.

2) : $p(x)$; the functional response of the predator y with respect to the prey x and $p(0) = 0, p'(x) > 0$ for all $x \geq 0$.

3) : $q(x)$; the functional response of the predator z with respect to the prey x and $q(0) = 0$, $q'(x) > 0$ for all $x \geq 0$.

(H4) : we assume $q(x) = \alpha p(x)$, where α is a positive constant.

5) : $g(y)$; density-dependent death rate of the predator y and

$g(y)$; $g'(y) \geq 0$ for all $y > 0$.

(H6) : $h(z)$; density-dependent death rate of the predator z and

$h(0) > 0$, $h'(z) \geq 0$ for all $z > 0$.

Clearly the system (2.1) has equilibiruim point $E_0(0, 0, 0)$. By assumption (H1), $E_1(k, 0, 0)$ is also an equilibrium point. We assume that each of the predators y and z can survive on the prey x, that is there exist equilibirium points $\overset{\bullet}{E}\left(\overset{\bullet}{x},\overset{\bullet}{y},0\right)$ and $\hat{E}(\hat{x},0,\hat{z})$ such that

$$\overset{\bullet}{x} f\left(\overset{\bullet}{x}\right) - \overset{i}{y}\, p\left(\overset{\bullet}{x}\right) = 0 \tag{2.2}$$

$$-g\left(\overset{\bullet}{y}\right) + c_1 p\left(\overset{\bullet}{x}\right) = 0$$

and

$$\hat{x} f(\hat{x}) - \hat{z} q(\hat{x}) = 0$$

$$-h(\hat{z}) + d_1 q(\hat{x}) = 0 \tag{2.3}$$

where $\overset{\bullet}{x},\overset{\bullet}{y},\hat{x},\hat{z} > 0$ and $\overset{\bullet}{x} < k, \hat{x} < k$ · (2.4)

GLOBAL ASYMPTOTIC STABILITY OF EQUILIBRIA $\overset{\bullet}{E}\left(\overset{\bullet}{x},\overset{\bullet}{y},0\right)$ and $\hat{E}(\hat{x},0,\hat{z})$.

3.1 Assume (H1)-(H6) hold for the system (2.1) and in a neighbourhood of $\left(\overset{\bullet}{x},\overset{\bullet}{y},0\right)$ in the positive cone, the function $\frac{xf(x)}{p(x)}$ is strictly decreasing.

Then the equilibrium point $\overset{*}{E}\left(\overset{*}{x},\overset{*}{y},0\right)$ is globally asymptotically stable.

Proof :

Define a Lyapunov function $V(x,y,z)$ as [see [1]] :

$$V(x,y,z)=\int_{x^*}^{x}\left[c_1\left(1-\frac{p(x^*)}{p(w)}\right)+d_1\left(1-\frac{q(x^*)}{q(w)}\right)\right]dw+\int_{y^*}^{y}\left(\frac{w-y^*}{w}\right)dw+z \qquad (3.1)$$

Now $\left(\overset{*}{x},\overset{*}{y},0\right)=0$ and due to (H2) and (H3), (*x*, *y*, *z*) is positive in the region:

$0<x<\overset{*}{x}<k,\ 0<\overset{*}{y}<y<\beta_1,\ 0<z<\beta_2,\ \beta_1,\ \beta_2$ are positive constants.

$$\frac{d}{dt}V(x,y,z)=c_1\left[p(x)-p\left(\overset{*}{x}\right)\right]\left[\frac{xf(x)}{p(x)}-\frac{zq(x)}{p(x)}-y\right]$$

$$+d_1\left[q(x)-q\left(\overset{*}{x}\right)\right]\left[\frac{xf(x)}{q(x)}-\frac{yp(x)}{q(x)}-z\right]$$

$$+\left(y-\overset{*}{y}\right)\left[-g(y)+c_1p(x)\right]+z\left[-h(z)+d_1q(x)\right]$$

using (2.2), (H4) and with some algebraic manipulations we get

$$\dot V(x,y,z)=\left[c_1\left(p(x)-p\left(\overset{*}{x}\right)\right)+\frac{1}{\alpha}d_1\left(q(x)-q\left(\overset{*}{x}\right)\right)\right]\left[\frac{xf(x)}{p(x)}-\frac{\overset{*}{x}f\left(\overset{*}{x}\right)}{p\left(\overset{*}{x}\right)}\right]$$

$$+\left(y-\overset{*}{y}\right)\left[\left(g\left(\overset{*}{y}\right)-g(y)\right)+\frac{d_1}{\alpha}\left(q\left(\overset{*}{x}\right)-q(x)\right)\right]$$

$$+z\left[\alpha c_1\left(p\left(\overset{*}{x}\right)-p(x)\right)+\left(h(0)-h(z)\right)\right]<0$$

Thus $\overset{*}{E}\left(\overset{*}{x},\overset{*}{y},0\right)$ is globally asymptotically stable.

3.2 Assume (H1)-(H6) hold for the system (2.1) and in a neighbourhood of $(\hat{x},0,\hat{z})$ in the positive cone, the function $\frac{xf(x)}{p(x)}$ is strictly decreasing. Then the equilibrium point $\hat{E}(\hat{x},0,\hat{z})$ is globally asymptotically stable.

Proof :

Define Lypunov function (x, y, z) as :

$$V(x,y,z)=\int_{\hat{x}}^{x}\left[c_1\left(1-\frac{p(\hat{x})}{p(w)}\right)+d_1\left(1-\frac{q(\hat{x})}{q(w)}\right)\right]dw+y+\int_{\hat{z}}^{z}\left(\frac{w-\hat{z}}{w}\right)dw \tag{3.2}$$

Rest of the proof follows as in Lemma 3.1.

Remark 3.1.

We consider equilibirium points $E_0(0,0,0)$ and $E_1(k,0,0)$.

The eigenvalues of the variational matrix (E_0) of the system (2.1.) about E_0 (0,0,0) are:

$$\lambda_1=f(0)>0,\ \lambda_2=-g(0)<0,\ \text{and}\ \lambda_3=-h(0)<0. \tag{3.3}$$

Clearly $E_0(0,0,0)$ is a hyperbolic point and is unstable along the *x-axis*. This implies that the prey population x grows near E_0.

The eigenvalues of the variational matrix (E_1) of the system (2.1) about $E_1(k,0,0)$ are :

$$\lambda_1=kf'(k)<0,\ \lambda_2=-g(0)+c_1p(k),\ \lambda_3=-h(0)+d_1q(k) \tag{3.4}$$

Thus $E_1(k,0,0)$ is asymptotically stable along the *x-axis*. This implies that the prey population x remains in neighbourhood of k.

Remark 3.2.

For existence of $E\left(x,y,0\right)$ and $\hat{E}(\hat{x},0,\hat{z})$ it is necessary that

$$g(0)+c_1p(k)>0 \text{ and } -h(0)+d_1q(k)>0 \tag{3.5}$$

as it implies increase of predator population y and predator population z.

PERSISTENCE CRITERIA

In section 3, we have given necessary conditions for existence of equilibria $\overset{*}{E}\left(\overset{*}{x},\overset{*}{y},0\right)$ and $\hat{E}(\hat{x},0,\hat{z})$, and creiteria for their global asymptotic stability.

In this section, we shall assume global stability of these equilibria and obtain persistence criteria for the system (2.1.).

first, we prove the following two lemmas.

Lemma 4.1. The equilibrium $\overset{*}{E}\left(\overset{*}{x},\overset{*}{y},0\right)$ in the interior of the x-y plane is unstable in the positive direction orthogonal to x-y plane if

$$-h(0)+d_1q\left(\overset{*}{x}\right)>0 \text{ or } -h(0)+d_1\alpha p\left(\overset{*}{x}\right)>0.$$

Proof:

The proof is immediate upon computing the variational matrix $\left(\overset{*}{E}\right)$ of system (2.1) about $\overset{*}{E}\left(\overset{*}{x},\overset{*}{y},0\right)$.

We have :

$$V\left(\overset{\bullet}{E}\right)=\begin{pmatrix} f\left(\overset{\bullet}{x}\right)+\overset{\bullet}{x}f'\left(\overset{\bullet}{x}\right)-\overset{\bullet}{y}p'\left(\overset{\bullet}{x}\right) & -p\left(\overset{\bullet}{x}\right) & -q\left(\overset{\bullet}{x}\right) \\ -gc_1\overset{\bullet}{y}p'\left(\overset{\bullet}{x}\right) & \left(\overset{\bullet}{y}\right)+c_1p\left(\overset{\bullet}{x}\right)-\overset{\bullet}{y}g'\left(\overset{\bullet}{y}\right) & 0 \\ 0 & 0 & -h(0)+d_1q\left(\overset{\bullet}{x}\right) \end{pmatrix}$$

This if $-h(0)+d_1q\left(\overset{\bullet}{x}\right)>0$ or $-h(0)+d_1\alpha p\left(\overset{\bullet}{x}\right)>0$.

we have the result.

Lemma 4.2. The equilibrium $\hat{E}(\hat{x},0,\hat{z})$ in the interior of the x-z plane is unstable in the positive direction orthogonal to x-z plane if $-g(0)+c_1p(\hat{x})>0$.

Proof : The proof is immediate upon computing the variational matrix $V\left(\hat{E}\right)$ of the system (2.1) about $\hat{E}(\hat{x},0,\hat{z})$.

Now, to apply persistence criteria to our system (2.1), we have to check hypotheses (B1)-(B-4) of theorem 4.1. in Freedman and Waltman [4] and boundedness of solutions.

we have :

$$F(x,y,z)=f(x)-y\frac{p(x)}{x}-z\frac{q(x)}{x}$$

$$G_1(x,y,z)=-g(y)+c_1p(x)$$

$$G_2(x,y,z)=-h(z)+d_1q(x),$$

therefore; condition (B1) is trivially satisfied due to (H1)-(H6). Also notice that

$$p(0)=0,\ \text{and}\ p'(x)>0$$

implies $p(x)$ is a strictly increasing positive function. Similarly $q(x)$.

Condition (B2) is true due to (H1).

$$F(0,0,0)=f(0)>0,\ F(k,0,0)=f(k)=0$$

$$\frac{\partial f}{\partial x}(x,0,0)=f'(x)\le 0.$$

Satisfying (B3) there are no equilibria on *y-axis* or *z-axis* in y-z plane, for if we suppose that there exists an equilibrium $E(0, y_1, z_1)$ in y-z plane which is given by : $g(y_1) = 0$ and $h(z_1) = 0$, then this contradicts (H5) and (H6), and satisfying (B4) each predator can survive on prey, that is, there exist points $E\left(x, y, 0\right)$ and $\hat{E}(\hat{x}, 0, \hat{z})$ such that (2.2) and (2.3) hold.

Also see remark 3.2. Moreover, we require;

5) : Boundedness of solutions of system (2.1.). We suppose that the functions $f(x)$, $p(x)$, $q(x)$, $g(y)$ and $h(z)$ are sufficiently smooth so that the solution to the initial value problem (2.1) exists, is unique and continuable for positive values of t. Regarding boundedness of solution, see Freedman and Waltman [3] and [4].

We state and prove the main theorem.

Theorem 4.1 Let (B1)-(B5) hold. Let $\frac{xf(x)}{p(x)}$ be a strictly decreasing function and

$$-g(0) + c_1 p(\hat{x}) > 0 \text{ and } -h(0) + d_1 \alpha p\left(x\right) > 0 \tag{4.1}$$

Then the system (2.1.) persists.

Proof : By (B5) solutions are bounded.

By remark 3.1 the equilibrium $E_0(0,0,0)$ is unstable along the *x-axis* and unstable manifold of $E_1(k,0,0)$ is two dimensional. Conditions (4.1) follow from lemmas 4.1. and 4.2. Non-existence of limit cycles follows from lemmas 3.1. and 3.2.

This completes the proof.

Remark 4.1.

From (H4), we have

$$\alpha = \frac{q(x)}{p(x)} = \frac{\text{rate of prey consumption per predator } z \text{ at prey density } x}{\text{rate of prey consumption per predator } y \text{ at prey density } x}$$

Thus, if we consider α as a parameter then, the system (2.1.) will persit provided

$$\alpha > \frac{h(0)}{d_1 p\left(\overset{\bullet}{x}\right)}$$

Remark 4.2.

We have discussed persistence criteria for a system modeling the interaction of two competing predator populations living exclusively on a common prey. But in the same way, the persistence criteria can be obtained for the system modeling interactions between two prey populations and one predator population, that is,

$$\begin{aligned} \dot{x} &= xf(x) - zq(x) \\ \dot{y} &= yg(y) - zr(y) \qquad\qquad (4.2.) \\ \dot{z} &= z[-h(z) + d_1 q(x) + d_2 r(y)] \end{aligned}$$

and for three-level food web, that is

$$\begin{aligned} \dot{x} &= xf(x) - yp(x) - zq(x) \\ \dot{y} &= y[-g(y) + c_1 p(x)] - zr(y) \qquad\qquad (4.3) \\ \dot{z} &= z[-h(z) + d_1 q(x) + d_2 r(y)] \end{aligned}$$

For construction of Lyapunov functions for the systems (4.2) and (4.3) see [1].

Example 4.1.

To illustrate the theorem 4.1., consider the system with the Holling type II functional response.

$$\begin{aligned}\dot{x} &= x(1-x) - y\frac{2x}{1+x} - z\alpha\frac{2x}{1+x}\\ \dot{y} &= y\left[-(1+y)+\frac{33}{16}\frac{2x}{1+x}\right]\\ \dot{z} &= z\left[-(1+z)+\frac{11}{9}\alpha\frac{2x}{1+x}\right]\end{aligned} \tag{4.4}$$

Here, $k = 1, \dfrac{xf(x)}{p(x)} = \dfrac{1-x^2}{2}$

Thus $\dfrac{xf(x)}{p(x)}$ is a strictly decreasing function for x>0.

$$\overset{\bullet}{E}\left(\overset{\bullet}{x},\overset{\bullet}{y},0\right) = \overset{\bullet}{E}\left(\frac{1}{2},\frac{3}{8},0\right)$$

$$p\left(\overset{\bullet}{x}\right) = p\left(\frac{1}{2}\right) = \frac{2}{3}$$

$$\alpha > \frac{h(0)}{d_1 p\left(\overset{\bullet}{x}\right)} = \frac{27}{22}. \text{ Thus take } \alpha = 2$$

$$\hat{E}(\hat{x},0,\hat{z}) = \hat{E}\left(\frac{1}{3},0,\frac{2}{9}\right)$$

To check condition (4.1)

$$-g(0) + c_1 p(\hat{x}) = -1 + \frac{33}{16}\times\frac{1}{2} = \frac{1}{32} > 0$$

$$-h(0)+d_1\alpha p\left(\overset{\bullet}{x}\right)=-1+\frac{11}{9}\times 2\times\frac{2}{3}=-1+\frac{44}{27}=\frac{17}{27}>0$$

Theorem 4.1. applies and hence the system (4.4.) is persistent.

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ON KÖTHE-TOEPLITZ DUAL OF A CERTAIN SEQUENCE SPACE AND MATRIX-TRANSFORMATIONS

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ABSTRACT

We extend the well known sequence space $l(p)$ to $l(p,\upsilon)$ for $\upsilon \geq 0$ and find its Köthe-Toeplitz dual and continuous dual. Some matrix transformations have also been considered.

INTRODUCTION

Let l_∞ and c be the Banach spaces of bounded and convergent sequences $x = (x_k)$ with usual norm $\|x\| = \sup_k |x_k|$. For $p = (p_k)$ with $p_k > 0$ and $\sup p_k < \infty$ Maddox [1] defined the space

$$l(p) = \left\{ x = (x_k) : \sum_k |x_k|^{p_k} < \infty \right\};$$

and it was shown that the continuous dual of $l(p)$, i.e. $l^*(p)$ could be identified with $l(q)$, where $(1/p_k) + (1/q_k) = 1$

Further, in [2] the Köthe-Toeplitz dual of $l(p)$ was determined.

In this paper, we extend the space $l(p)$ to $l(p,\upsilon)$ for $\upsilon \geq 0$ and determine the continuous and Köthe-Toeplitz duals of $l(p,\upsilon)$. We also characterize the matrices of classes $(l(p,\upsilon), l_\infty)$ and $(l(p,\upsilon), c)$.

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Definition :

1.1 For $p=(p_k)$ with $p_k>0$ and sup $p_k<0$, we define for $\upsilon \geq 0$.

$$l(p,\upsilon)=\left\{x=(x_k) : \sum_k k^{-\upsilon}|x_k|^{p_k}<\infty\right\}$$

where $l(p)=\left\{x=(x_k) : \sum_k |x_k|^{p_k}<\infty\right\}$

and $$M(q,\upsilon)=\left\{a=(a_k) : \sum_k k^{\upsilon(q_k^{-1})}|a_k|^{q_k} E^{-q_k/p_k}<\infty\right\}$$

where $E>1$ and $1/p_k+1/q_k=1$

1.2 Paranormed Space :

A paranorm g defined on a linear space X, is a function $g : X \rightarrow R$ having the following properties.

(i) $g(\theta)=0$ where θ is the zero element in X.

(ii) $g(x)=g(-x)$ for all $x \in X$.

(iii) $g(x+y)\leq g(x)+g(y)$ for all $x, y \in X$.

(iv) The scalar multiplication is continuous, that is $\lambda_n \rightarrow \lambda(n\rightarrow\infty)$ &

$g(x_n-x)\rightarrow 0$ as $n\rightarrow\infty$ for, λ_n, $\lambda \in C$ and x_n, $x \in X$ .imply that

$g(\lambda_n x_n-\lambda x)\rightarrow 0(n\rightarrow\infty)$.

A paranormed space is a linear space X together with a paranorm g. A total paranorm is a paranorm such that $g(x)=0$ implies $x=0$.

It is easy to see that $l(p,\upsilon)$ is paranormed space by

$$h(x) = \left[\sum k^{-\upsilon} |x_k|^{p_k}\right]^{\frac{1}{M}}$$

where $M = \max(1, \sup p_k)$.

For; (i) $h(x) = \left[\sum k^{-\upsilon} |x_k|^{p_k}\right]^{\frac{1}{M}}$

$= 0$ if and only if $x = 0$

(ii) $$h(-x) = \left[\sum k^{-\upsilon} |-x_k|^{p_k}\right]^{\frac{1}{M}}$$

$$= \left[\sum k^{-\upsilon} |x_k|^{p_k}\right]^{\frac{1}{M}}$$

$$= h(x)$$

(iii) $$h(x+y) = \left[\sum k^{-\upsilon} |x_k + y_k|^{p_k}\right]^{\frac{1}{M}}$$

By minkowski's inequality we have

$$h(x+y) \leq \left[\sum k^{-\upsilon} |x_k|^{p_k}\right]^{\frac{1}{M}} + \left[\sum k^{-\upsilon} |y_k|^{p_k}\right]^{\frac{1}{M}}$$

$$= h(x) + h(y)$$

Hence $h(x+y) \leq h(x) + h(y)$

(iv) If λ_n, $\lambda_0 \in C$ with $\lambda_n \to \lambda_0$ $(n \to \infty)$ and if x_n, $a \in l(p,\upsilon)$ with $x_n \to a$ $(n \to \infty)$ in the sense that $h(x_n - a) \to 0 (n \to \infty)$

& $$h(\lambda_n x_n - \lambda_n a) = \left[\sum_n n^{-\upsilon} |\lambda_n x_n - \lambda_0 a|^{p_n}\right]^{\frac{1}{M}}$$

$$= \left[\sum_n n^{-\upsilon} \left| (\lambda_n - \lambda_0)(x_n - a) + \lambda_0 (x_n - a) + (\lambda_n - \lambda_0) a \right|^{p_n} \right]^{\frac{1}{M}}$$

$$\leq \left[\sum n^{-\upsilon} \left| (\lambda_n - \lambda_0)(x_n - a) \right|^{p_n} \right]^{\frac{1}{M}} + \left[\sum n^{-\upsilon} \left| \lambda_0 (x_n - a) \right|^{p_n} \right]^{\frac{1}{M}}$$

$$+ \left[\sum n^{-\upsilon} \left| (\lambda_n - \lambda_0) a \right|^{p_n} \right]^{\frac{1}{M}}$$

Now $\left| (\lambda_0) \right|^{p_n} \leq \max(1, |\lambda_0|) = H(\text{say})$

Hence, $h(\lambda_n x_n - \lambda_0 a) \to 0 (n \to \infty)$

So, $l(p,\upsilon)$ is a total paranormed space.

In the next section we prove that $l^*(p,\upsilon) = M(q,\upsilon)$ i.e. the Köthe-Toeplitz dual of $l(p,\upsilon)$ is $M(q,\upsilon)$; and $l^*(p,\upsilon)$ is isormorphic to $M(q,\upsilon)$.

CONTINUOUS AND KÖTHE-TOEPLITZ DUALS

If (X, h) is a paranormed space with paranorm h then we denote X^* the continuous dual of X that is the set of all continuous linear functionals on X. If E is a set of complex sequences $x = (x_k)$ then E^+ will denote the generalized Köthe-Toeplitz dual of E.

$$E^+ = \left\{ a = (a_k) : \sum_k a_k x_k \text{ converges for all } x \in E \right\}$$

Theorem :

2.1 Let $1 < p_k \leq \sup p_k < \infty$ for all k. Then for $\upsilon \geq 0$ $l^*(p,\upsilon) = M(q,\upsilon)$.

Proof :

Here we use the same analysis as in [2]. Let $x \in l(p,\upsilon)$ and $a \in M(q,\upsilon)$. Therefore

$$\sum_k |a_k x_k| \le \sum_k |a_k| k^{\upsilon/p_k} . |x_k| k^{-\upsilon/p_k}$$

$$\le \sum_k |a_k|^{q_k} k^{\upsilon q_k/p_k} . E^{-q_k/p_k} + E\sum_k |x_k|^{p_k} k^{-\upsilon}$$

$$\le \sum_k |a_k|^{q_k} k^{\upsilon(q_k-1)} E^{-q_k/p_k} + E\sum |x_k|^{p_k} k^{-\upsilon}$$

$$< \infty$$

Hence $\sum_k a_k x_k$ is convergent. Therefore $a \in l^*(p,\upsilon)$.

Conversely, let $\sum_k a_k x_k$ converges and $x \in l(p,\upsilon)$. This implies that $a \in M(q,\upsilon)$, for otherwise we can determine integers.

$$0 = n(0) < n(1) < n(2) < \dots\dots\dots$$

such that

$$M_j = \sum_{k=n(j-1)+1}^{n(j)} k^{\upsilon(q_k^{-1})} |a_k|^{q_k} (j+1)^{-q_k/p_k} > 1$$

Define a sequence $x = (x_k)$ by

$$x_k = (sign\ a_k)|a_k|^{q_k^{-1}} (j+1)^{-q_k} M_j^{-1} k^{\upsilon(q_k^{-1})}, \text{ for } n(j-1) \le k \le n(j)$$

$$= 0, \text{ otherwise.}$$

Therefore

$$\sum_{k=n(j-1)+1}^{n(j)} k^{-\upsilon} |x_k|^{p_k} = \sum_{k=n(j-1)+1}^{n(j)} k^{-\upsilon} k^{\upsilon} (q_k^{-1})^{p_k} |a_k|^{(q_k^{-1})p_k} (j+1)^{-q_k/p_k} M_j^{-p_k}$$

$$\leq (j+1)^{-2} M_j^{-1} \sum_{k=n(j-1)+1}^{n(j)} k^{-q_k^{-\upsilon} k^{\upsilon} (q_k^{-1})^{p_k}} |a_k|^{q_k} (j+1)^{-q_k/p_k}$$

$$\leq (j+1)^{-2}$$

i.e. $x \in l(p,\upsilon)$. But

$$\sum_{k=n(j-1)+1}^{n(j)} a_k x_k = (j+1)^{-1}$$

So that $\sum_k a_k x_k$ diverges. Hence a contradictioin and therefore $a \in M(q,\upsilon)$.

Theorem :

2.2 Let $1 < p_k \leq \sup p_k < \infty$. Then $l^*(p,\upsilon)$ is isomorphic to $M(q,\upsilon)$.

Proof :

It is easy to see that $x \in (p,\upsilon)$ can be written as $x = \sum_k x_k e_k$ where $e_k = (0,0,\ldots,$ $0,1,0,\ldots)$. Therefore for any $f \in l^*(p,\upsilon)$.

$f(x) = \sum_k x_k f(e_k) = \sum_k x_k a_k$, where $f(e_k) = a_k$.

By theorem 2.1, $a \in M(q,\upsilon)$ if $\sum_k a_k x_k$ converges for $x \in l(p,\upsilon)$.

Now, if $x \in l(p,\upsilon)$ and $a \in M(p,\upsilon)$; $\sum_k a_k x_k$ converges and defines a linear functional on $l(p,\upsilon)$. Furthermore, it is easy to check that

$$\sum_{k=1}^{\infty} |a_k x_k| \leq E\left(\sum k^{\upsilon(q_k^{-1})} |a_k| E^{\frac{-q_k}{p_k}} + 1 \right) g(x)$$

whenever $g(x) \le 1$. Therefore $\sum_k a_k x_k$ defines an element of $l^*(p,\upsilon)$. Obviously, the map $T: l^*(p,\upsilon) \to M(q,\upsilon)$ given by $T(f) = (a_1, a_2.....)$ is linear and bijective. Hence $l^*(p,\upsilon)$ is isomorphic to $M(q,\upsilon)$.

MATRIX TRANSFORMATIONS

Let $A = (a_{nk})$ be an infinite matrix of complex numbers $a_{nk}(n,k = 1,2,.....)$ and X, Y be two subsets of the space of complex sequences. We say that the matrix A defines a matrix transformation from X into Y, if for every sequence $x = (x_k) \in X$ the sequence $Ax = (y_n) \in Y$, where $Y_n = \sum a_{nk} x_k$ provided the series is convergent. The class of all such matrix transformations from X into Y will be denoted by (X, Y).

In this section we characterize the matrices of the classes $(l(p,\upsilon), l_\infty)$ and $(l(p,\upsilon), c)$.

Theorem :

3.1 Let $1 < p_k \le \sup p_k < \infty$. Then $A \in (l(p,\upsilon), l_\infty)$ if and only if there exists an integer $E > 1$ such that $U_n(E) < \infty$ for every n, where

$$U_n(E) = \sum_k k^{\upsilon(q_k^{-1})} |a_{nk}|^{q_k} E^{-q_k/p_k} < \infty .$$

Proof :

Necessity : Let $A \in (l(p,\upsilon), l_\infty)$ and $x \in l(p,\upsilon)$. Therefore $\sum_k a_{nk} x_k < \infty$ for each n and for $x \in l(p,\upsilon)$. Hence $(a_{nk})_k \in l^*(p,\upsilon)$ for each n. Since $l^*(p,\upsilon)$ is isomorphic to $l^*(p,\upsilon)$. Therefore $A_n(x) = \sum a_{nk} x_k$ is an element of $l^*(p,\upsilon)$. We know that $l(p,\upsilon)$ and l_∞ both are complete and by wilansky [3] that there exists a constants K such that

$$\|A_n(x)\| \leq K\|x\|$$

Hence by the principle of condensation of singularities [4]

$U_n(E) < \infty$, for every n.

Sufficiency : Let $x \in l(p,\upsilon)$ and the condition $U_n(E) < \infty$ hold for every n. Therefore, for $E > 1$.

$$\sum_k |a_{nk}x_k| \leq \sum_k k^{\upsilon(q_k^{-1})} |a_{nk}|^{q_k} E^{-q_k/p_k} + E\sum_k |x_k|^{pk} k^{-\upsilon} < \infty$$

Hence $A \in (l(p,\upsilon), l_\infty)$.

This completes the proof of the theorem.

Theorem :

3.2 Let $1 < p_k \leq \sup p_k < \infty$. Then $A \in (l(p,\upsilon), c)$ if and only if

(i) for $E > 1$. $U_n(E) < \infty$ for every n; and

(ii) $a_{nk} \to \alpha_k (n \to \infty)$ for every k.

Proof :

Necessity : Condition (i) follows as in Theorem 3.1; and condition (ii) follows from the fact that $e_k \in l(p,\upsilon)$;

Sufficiency : Let the conditions (i) and (ii) hold and that $x \in l(p,\upsilon)$. Therefore as in Thbeorem 3.1, we have

$$\sum |a_{nk}x_k| < \infty,$$

i.e. $\sum_k a_{nk} x_k$ is absolutely convergent for

each n and for $x \in l(b,\upsilon)$. Moreover, we can choose an integer $k_0 > 1$ such that

$$(1) \quad g_{k_0}(x) = \sum_{k=k_0}^{\infty} k^{-\upsilon} |x_k|^{p_k} < 1$$

Therefore by theorem 2.2 we have

$$(1) \quad \sum_{k \geq k_0} |a_{nk} - d_k| |x_k| \leq E \left(\sum_{k \geq k_0} k^{\upsilon(q_k^{-1})} |a_{nk} - \alpha_k|^{q_k} E^{-q_k/p_k} + 1 \right) (g_{k_0}(x))^{1/M}$$

where $M = \max(1, \sup p_k)$. Clearly (1) holds if $g_{k_0}(x) = 0$.

Since

$$\sum_{k \geq k_0} k^{\upsilon(q_k^{-1})} |a_{nk} - \alpha_k|^{q_k} E^{-q_k/p_k} \leq 2U_n(E) < \infty$$

Therefore by (1) we have

$\sum_k a_{nk} x_k \to \sum_k \alpha_k x_k (n \to \infty)$, and hence $A \in (l(p,\upsilon), c)$.

This completes the proof of the theorem.

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INITIAL VALUE ABELIAN THEOREM FOR DISTRIBUTIONAL MODIFIED STIELTJES TRANSFORMATIONS

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ABSTRACT

In the first part we define the space M'(r) and modified Stielfjes transformation introduced by Lavoiene & Misra [1] and Marichev [8] respectively. In the second part we use quasi-asymptotic behaviour of distribution to prove initial value Abelian theorem.

Key Words and Phrases : Modified Stieltjes transformation of distribution, quasi-asymptotic behaviour of of distributions.

AMS Subject Classification Codes (1991) : 44 and 46 F.

INTRODUCTION

The notion of quasi-asymptotic behaviour of tempered distributions was introduced by Zavialov [4]. Here we give, first several assertions concerning the quasi-asymptotic behaviour at 0^+ for elements from S'_+. The motivation for our investigation made in the first part are theorems of Abelian type for the distributional modified Stieltjes transformation, while we shall see in the second part that the quasi-asymptotic behaviour at 0^+ is more convenient than the ordinary asymptotic behaviour of a disbritution especially to prove Abelian type theorems for the modified Stieltjes transformation.

NOTATIONS AND DEFITIONS

N is the set of natural numbers and $N_0 = N \cup \{0\}$. R is the Set of real

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numbers and $¢$ is the set of complex numbers. S'_+ is the space of temperecd distributions with support in the $(0,\infty)$. S' is the space of distribution of slow growth or tempered distributions which is the dual of S. We set $R_+ = \{x \in R{:}x > 0\} = (0,\infty)$ and its closure in R is denoted by R_+^-, $R_-^- = (-\infty,0]$. We denote the partial differential operator by $D^p, p \in N_0$.

Definition 1[3] :

A function ρ : $(a,\infty) \to R$, $a \in R$ is called regularly varying function at infinity if it is positive, measurable and there exists a real number α such that for each $x > 0$

$$\lim_{k\to\infty} \frac{\rho(kx)}{\rho(k)} = x^\alpha .$$

the number α is called the index of ρ.

Definition 2 :

If function L is regularly varying function with index $\alpha = 0$ then it is called slowly varying function at infinity. Thus L is the positive, measurable on (a,∞) from definition 1, we have

$$\lim_{k\to\infty} \frac{L(kx)}{L(k)} = 1 .$$

Definition 3 (Space $M'(r)$) :

Let $M'(r)$, $r \in IR \setminus (-IN)$ denote the space of all distributions $f \in S'_+(IR)$ such that $f = t^{-r} D^k F$, for some $k \in INo$ and locally integrable function F, $\sup F \subset [0,\infty)$ and

$$\int_{IR} /F(t)/(t+\beta)^{-r-l-k} dt < \infty \text{ for } \beta > 0$$

Definition 4 :

The Stieltjes transformation $S_r(f)(s), r \in IR\backslash(-IN)$ is complex valued function defined by

$$S_r(f(t))(s) = \int_0^\infty \frac{f(t)}{(s+t)^{r+1}}\,dt \tag{1}$$

$s \in ¢\backslash(-\infty,0]$, $0 < t < \infty$, $r \in IR\backslash(-IN)$

Modified Stieltjes transformation introduced by Marichev [8] is

$$T_\alpha(f(x)) = \frac{1}{\Gamma(\alpha)}\int_0^\infty \left(1+\frac{x}{y}\right)^{-\alpha} \frac{1}{y} f(y)\,dy \quad x \in ¢\backslash(-\infty,0], \quad 0 < y < \infty$$

It can be written as,

$$T_\alpha(f(x)) = \frac{1}{\Gamma(\alpha)}\int_0^\infty \frac{y^{\alpha-1} f(y)}{(x+y)^\alpha}\,dy. \tag{2}$$

By putting, $r = \alpha - 1$, $f(t) = y^{\alpha-1} f(y)$ in (1)
we get,

$$S_{\alpha-1}(y^{\alpha-1} f(y))(x) = \int_0^\infty \frac{y^{\alpha-1} f(y)}{(x+y)^\alpha}\,dy. \tag{3}$$

By equation (2) & (3)

$$T_\alpha(f)(x) = \frac{1}{\Gamma(\alpha)} S_{\alpha-1}(y^{\alpha-1} f(y))(x).$$

By interchanging x by z & α by $r+1$

$$\Gamma(r+1) T_{r+1}(f)(z) = S_r(y^r f)(z). \tag{4}$$

Definiton 5 : Modified Stieltjes transformation T_{r+1}

Let us define the T_{r+1} - transformation $r \in IR\backslash(-IN)$. Assume that $f \in M'(r)$ and that $f = t^{-r}D^k F$ and $\int_{IR} |F(t)|(t+\beta)^{-r-1-k} < \infty$ for $\beta > 0$, holds. Now definition is as follows

$T_{r+1}(f)$, $r \in IR\backslash(-N)$ is complex valued function defined by

$$\Gamma(r+1)T_{r+1}(f)(s) = (r+1)_k \int_0^\infty \frac{F(t)}{(s+t)^{r+1+k}}\, dt \,.$$

$r \in IR\backslash(-lN)$, $s \in \not{c}\backslash(-\infty,0]$, $0 < t < \infty$

Definition 6 : Generalized modified Stieltjes transformation $\tilde{T}_{r+1}$

The $\tilde{T}_{r+1}$ - transformation of a distribution $f \in S'_+(IR)$ is complex valued function $\tilde{T}_{r+1}$ (f) defined by

$$\Gamma(r+1)\tilde{T}_{r+1}(f)(s) = \lim_{w\to\infty} < f(t),\ \eta(t)(s+t)^{-r-1}\exp(-wt) > \tag{5}$$

$w \in IR$, $s \in \Lambda \subset (\not{c}\backslash(-\infty,0])$, $\eta \in A(s)$.

where Λ is the set of complex number for which this limit exists and $A(s)$ is the family of all smooth functions, defined on R for which there exists $\varepsilon = \varepsilon_{\eta,s} > 0$ such that $0 \le \eta(t) \le 1$, $t \in R$, $\eta(t) = 1$ if t belongs to the ε - neighbourhood of $\bar{R}_+, \eta(t) = 0$ if it belongs to the complement of the 2 ε - neighbourhood of $\bar{R}_+$, where $\varepsilon > 0$ is arbitrary if $IMS \neq 0$, and $0 < 2\varepsilon < \max \mathrm{Re}s$, if for some $IMS = 0$ and $|D^p\eta(t)| \le C_p$,

$t \in IR$. If $\eta(t) \in A(s)$, $s \in (\not{c}\backslash R_-)$

$(R_- = (-\infty,0])$, then $\eta(t)(s+t)^{-r-1}\exp(-wt) \in S(IR)$ for $w \in R_+$, $r \in R$

INITIAL VALUE ABELIAN THEOREM

To prove this theorem we use the following definitions, lemma and propositions.

Definition 3.1 ([2]) :

Let $M'(r),\ r \in IR\setminus(-IN)$, denote the space of distributions f having supports in $\overline{IR}_+$ and admitting decomposition $f = t^{-r}\left(S + D^k F\right)$, where F is a function with support in the set $B = [\alpha,\infty),(\alpha \in IR_+),\ F(t)t^{-r-k-1}$ is integrable and S is distribution having support in $\overline{IR}_+ \setminus \operatorname{int} B$.

Now if $f = t^{-r}\left(S + D^p F\right) \in M'(r)$ the modifeid Stieltjes transformation of f defined by Lavoine & Misra [1] is

$$\Gamma(r+1)(T_{r+1}f)(s) =< S(t),\ (s+t)^{-r-1} > + \frac{\Gamma(r+q+1)}{\Gamma(r+1)} \int_{\alpha}^{\infty} F(t)(s+t)^{-r-q-1} dt \quad (3.1)$$

If $f \in M'(r_0)$, then we can prove that for all $r \geq r_0$, modified Stieltjes transform $\Gamma(r+1)\left(\widetilde{T}_{r+1}f\right)(s)$ given by definition of modified Stieltjes equation (5) exists and

$$\Gamma(r+1)\left(\widetilde{T}_{r+1}f\right)(s) = \Gamma(r+1)\widetilde{S}_{r+1}(s) + (r+1)_q \int_{0}^{\infty} \frac{F(t)}{(s+t)^{r+q+1}} dt,\ s \in \left(\text{¢} \setminus \overline{R}_-\right) \quad (3.2)$$

For a distribution $S \in (S_+')$ with support in in a compact $k \subset \overline{IR}_+$, the modified Stieltjes transform is

$$\Gamma(r+1)\widetilde{S}_{r+1}(s) = \lim_{w \to 0} < S(t),\ \frac{\eta(t)\exp(-wt)}{(s+t)^{r+1}} >=< S(t),\ \frac{\xi(t)}{(s+t)^{r+1}} > \quad (3.3)$$

where, $\xi \in C^{\infty}$, $\operatorname{supp}\xi \subset k^{3,\varepsilon}$ & $\xi(t) = 1,\ t \in k^{\varepsilon}$, k^{ε} is the ε – neighbourhood of k.

Definition 3.2 ([2]) :

Distribution $f \in S'_+$ has the quasi-asymptotic behaviour at 0^+ related to $(1/k)^a\ L(1/k)$ with the limit $g \in S'_+$, $g \neq 0$, if for every $\phi \in S$

$$\lim_{k\to\infty} \frac{k^a}{L(1/k)} < f(t/k),\ \phi(t) >=< g(t),\ \phi(t) >. \tag{3.4}$$

We write in short $f \overset{q}{\sim} g$ at 0^+, related to $(1/k)^a L(1/k)$.

Proposition 3.1 :

Let f_1 & f_2, belong to S'_+ & that $f_1 = f_2$ in some neighbourhood of zero.

If $f_1 \overset{q}{\sim} g$ at 0^+, related to $(1/k)^b L(1/k)$, then $f_2 \overset{q}{\sim} g$ at 0^+, related to $(1/k)^b L(1/k)$ too.

Theorem 3.1 :

Let $f \in S'_+$ then, $f \overset{q}{\sim} g$ at 0^+ related to $(1/k)^b L(1/k)$ if and only if there exists $a \in R_+$, $a > -b-2$, such that $F = f_{a+1} * f \in L'_{loc}$, F defines a regular tempered distribution and

$F(t) \sim Cf_{b+a+2}(t)L(t)$, $t \to 0^+$. we know that

$$\text{Supp}\, F \subset IR_+ \ \&\ |\mathrm{F(t)}| \leq (1+\|t\|)^m t^a,\ t \in R_+,\ a > m. \tag{3.5}$$

Lemma 3.1 :

Let $r-b>0$, $b+p+1>0$. Then

$$\lim_{k\to\infty} \frac{k^{-(r-b)}}{L(1/k)} (r+1)_p \int_0^\infty \frac{f_{b+p+1}(t)L(t)}{((s/k)+t)^{r+p+1}} dt = \frac{\Gamma(r-b)}{\Gamma(r+1)} s^{-(r-b)}.$$

$L(t)$ is slowly varying at 0^+, $t \in IR$ $s \in$ ¢ $\setminus \overline{R}_-$.

Lemma 3.2 :

Let $G \in L'_{loc}(IR)$, supp $G \subset [0,\infty)$, $r > \alpha > -1$, $G(x) \sim x^{\alpha}$ as $x \to 0^{+}$ and

$$\int_0^\infty \left| \frac{G(t)}{(z+t)^{r+1}} \right| dt < \infty,\ z \in ¢ \setminus (-\infty, 0].$$

Then,

$$z \to z^{r-\alpha}\Gamma(r+1)(T_{r+1}G)(z)$$

is a bounded analytic function in the domain $\Omega_{0,\epsilon} \cap B(0,R)$, $0 < \varepsilon < \pi/2$, $R > 0$.

$\left(\Omega_{a,\epsilon} = \{a + \mathrm{Re}^{i\phi}; R > 0, -\pi + \epsilon \le \phi \le \pi - \epsilon\}, a \ge 0 \ \&\ B(0,R) = \{z : |z| < R\}\right)$.

Theorem 3.2 : Initial value theorem

Let $f \in M'(r_0)$ and that $f \overset{q}{\sim} g$ at 0^{+} related to $(1/k)^{b} L(1/k)$ then $g = CF_{b+1}$ & for s belonging to the closed domain

$$\Omega_{\epsilon} = \{s \in ¢,\ \arg s \in [-\pi + t, -\pi - \varepsilon],\ (0 < \varepsilon < \pi/4)\}.$$

We have

$$\lim_{k \to \infty} \frac{(s/k)^{r-b}\Gamma(r+1)(T_{r+1}f)(s/k)}{L(1/k)} = C\frac{\Gamma(r-b)}{\Gamma(r+1)} \tag{3.6}$$

where, $r \neq 1$, $r > b$, and $r \geq r_0$. If $L = 1$ the convergence is uniform in Ω_{ϵ}.

Proof :

By definition 3.1, $f = t^{-r}(S + D^{p}F)$ where supp$f \subset B = [\alpha, \infty)$, $(\alpha \in IR_{+})$. $F(x)x^{-r-q-1}$ is integrable and S is distribution having support in the set $k \in (IR_{+} \setminus B)$.

By proposition 3.1. $\overset{\vee}{S} \sim g$ at 0^+ related to $(1/k)^b L(1/k)$.

Hence, $S = t^{-r} D^p G$, $G \in L'_{loc}$, $G \sim CF_{b+p+1}$ as $t \to 0^+$ ($\because$ Theorem 3.1)

Let k^ϵ be the ε – neighbourhood of the compact set k and let the function ξ be from C^∞ such that $\xi(t) = 1$, $t \in k$, $\xi(t) = 0$, $t \notin k^{3\varepsilon}$; $|\mathrm{D}^\alpha \xi| \leq C_\alpha C(\varepsilon)$

Using Leibniz Formula, we have,

$$S(t) = \xi(t)S(t) = \xi(t)t^{-r}D^pG(t) = D^p\left(t^{-r}\xi(t)G(t)\right) - \sum_{\substack{\alpha \leq p \\ \alpha \neq p}} D^\alpha\left(\xi_\alpha(t)t^{-r}G(t)\right), \qquad (3.7)$$

Where, ξ_α are functions with their support in $k^{3\alpha} \setminus k^\varepsilon([\alpha, \alpha + 3\varepsilon])$

$$\left(\because t^{\alpha-1}D^kF = D^k\left(t^{\alpha-1}F\right) - \sum_{p \leq k} D^p\left(t_p^{\alpha-1}F\right)\right)$$

Let $s \in ¢$, $\arg s \neq \pi$. The modified Stieltjes transformation of the distribution f is

$$\Gamma(r+1)(T_{r+1}f)(s) \;=\; < S(t)\xi(t), \frac{\eta(t)}{(s+t)^{r+1}} > + (r+1)_q \int_\alpha^\infty \frac{F(t)}{(s+t)^{r+q+1}} \qquad \text{(by 3.1)}$$

$$= (r+1)_p \left[\int_0^\alpha \frac{G(t)}{(s+t)^{r+p+1}} + \int_\alpha^{\alpha+3\varepsilon} \frac{G(t)\xi(t)}{(s+t)^{r+p+1}} dt - \sum_{\substack{\alpha \leq p \\ \alpha \neq p}} \int_\alpha^{\alpha+3\varepsilon} \frac{\xi_\alpha(t)G(t)}{(s+t)^{r+p+1}} \right.$$

$$\left. + (r+1)_q \int_\alpha^\infty \frac{F(t)}{(s+t)^{r+q+1}} \right] \qquad \text{(by 3.7)}$$

Now by Lemma 3.1

$$\lim_{k \to \infty} \frac{k^{-(r-b)}}{L(1/k)} \Gamma(r+1)(T_{r+1}f)(s/k) - \frac{C\Gamma(r-b)}{\Gamma(r+1)} s^{-(r-b)}$$

$$=\lim_{k\to\infty}\frac{k^{-(r-b)}}{L(1/k)}(r+1)_p\int_0^{\alpha}\frac{G(t)-f_{b+p+1}(t)L(t)}{((s/k)+t)^{r+p+1}}dt+\lim_{k\to\infty}\frac{k^{-(r-b)}}{L(1/k)}\int_0^{\infty}H(s/k,t)dt \quad (3.8)$$

where, $H(s/k,t), t\in IR_+$ is function with the properties. $\lim_{k\to\infty}H(s/k,t)=H(0,t)$ and $H(0,t)\in L'$.

As $r>b$

$$\lim_{k\to\infty}\frac{k^{-(r-b)}}{L(1/k)}\int_0^{\infty}H(s/k,t)dt=0 \quad (3.9)$$

Now we have to prove that the first term of (3.8) tends to zero.

We have,

$$G(t)-Cf_{b+p+1}(t)L(t)=d(t)f_{b+p+1}(t)L(t), t\in K \quad (3.10)$$

where d is a locally integrable function and $d(t)\to 0$, when $t\to 0^+$.

Therefore we can choose k_0 in such a way that for any $\varepsilon>0, |d(t/k)|<\varepsilon, t\in k$ and $k\geq k_0$. Then

$$\left|\frac{k^{-(r-b)}}{L(1/k)}(r+1)_p\int_0^{\alpha}\frac{G(t)-f_{b+p+1}(t)L(t)}{((s/k)+t)^{r+p+1}}dt\right|$$

$$\leq\frac{k^{-(r-b)}}{L(1/k)}(r+1)_p\int_0^{\alpha}\frac{d(t)f_{b+p+1}(t)L(t)}{|((s/k)+t)|^{r+p+1}}dt$$

$$\leq\varepsilon\frac{(r+1)_p}{L(1/k)}\int_0^{\alpha}\frac{f_{b+p+1}(u)L(u/k)}{|s+u|^{r+p+1}}du \quad (3.11)$$

$\therefore$ We proved that the first term on the right hand side of (3.8) tends to zero when $k \to \infty$.

$\therefore$ Equation 3.8 becomes

$$\lim_{k\to\infty} \frac{k^{-(r-b)}}{L(1/k)} \Gamma(r+1)(T_{r+1}f)(s/k) - C\frac{\Gamma(r-b)}{\Gamma(r+1)} s^{-(r-b)} = 0 \qquad \text{(by (3.9) \& (3.11))}$$

$$\therefore \lim_{k\to\infty} \frac{k^{-(r-b)}}{L(1/k)} \Gamma(r+1)(T_{r+1}f)(s/k) = C\frac{\Gamma(r-b)}{\Gamma(r+1)} s^{-(r-b)}.$$

which is required result. (by result (3.6))

For the proof of last statement of theorem. We use lemma 3.2 & Theorem 3.1 it follows that $z^{r-b}\Gamma(r+1)(T_{r+1}f)(z)$, is a bounded analytic function in $\Omega_{0,\epsilon} \cap B(0,R)$. Let $M(z) = \Gamma(r+1)(T_{r+1}f)(1/z)$, $z \in ¢ \setminus (-\infty,0]$.

The function $w^{r-b}M(w), w \in \Omega_{0,\epsilon} \cap \{w:|w| > 1/R\}$, $\varepsilon > 0$, $R > 0$ is analytic and bounded. The first part of theorem 3.2 implies $x^{r-b}M(x) \to A$, as $x \to \infty$, where

$$A = \frac{C\Gamma(r-b)}{\Gamma(r+1)}.$$

This implies that the same holds for $w^{r-\alpha}M(w)$ in the domain $\Omega_{a,\epsilon} \cap \{w:|w| > 1/R\}$, $a > 0$, $\varepsilon > 0$, $R > 0$.

By the Montcl's theorem it follows that $w^{r-b}M(w)$ converges to A uniformly in this domain & thus $w^{r-b}M(w)$ converges uniformly to A in $\Omega_{0,\epsilon}$ as $|w| \to \infty$. This implies that

$$z^{r-b}\Gamma(r+1)(T_{r+1}f)(z) \to A \text{ uniformly in } \Omega_\epsilon \text{ as } z \to 0.$$

Hence the proof of theorem.

Lemma 3.3 :

Let $f \in M'(r)$. Then for $x \in IR_+$ and $s \in ¢ \setminus \overline{R}_-$

$$\Gamma(r+1)(T_{r+1}f)(sx) = s(r+1)\int_{x}^{\infty} \Gamma(r+2)(T_{r+2}f)(st)dt$$

Theorem 3.3 :

If $f \in M'(r)$ and $\Gamma(r+2)(T_{r+2}f)(x) \sim Cx^{-(r-b)-1}L(x)$, $x \to 0^+$, where $b < r$,

then $\Gamma(r+1)(T_{r+1}f)(x) \sim C\dfrac{(r+1)}{(r-b)}x^{-(r-b)}L(x)$, $x \to 0^+$.

Proof :

Let $\varepsilon > 0$ be fixed. There exists $\delta \in IR_+$ such that for $0 < x < \delta$.

$$\Gamma(r+2)(T_{r+2}f)(x) = C(1+ \in(x))x^{-(r-b)-1}L(x) \tag{3.12}$$

Where $\in$ is locally integrable function in the δ – neighbourhood of zero and such that

$|C\varepsilon(x)| \leq \varepsilon / 8$, $0 < x < \delta$, with $M = C(r-b)^{-1}$ and $N = (r+1)$.

We have from Lemma 3.2.

$$\left|\frac{\Gamma(r+1)(T_{r+1}f)(x)}{NMx^{b-r}L(x)} - 1\right| \leq \left|\int_{x}^{\delta}\frac{\Gamma(r+2)(T_{r+2}f)(t)dt}{Mx^{b-r}L(x)} - 1\right| + \left|\int_{\delta}^{\infty}\frac{\Gamma(r+2)(T_{r+2}f)(t)dt}{Mx^{b-r}L(x)}\right|.$$

Now there exist $\overline{\delta}$, $0 < \overline{\delta} < \delta$, such that for $0 < x < \overline{\delta}$

$$\left|\int_{\delta}^{\infty}\frac{\Gamma(r+2)(T_{r+2}f)(t)dt}{Mx^{b-r}L(x)}\right| \leq \varepsilon / 8.$$

Thus, for $0 < x < \overline{\delta}$ we have

$$\left|\frac{\Gamma(r+1)(T_{r+1}f)(x)}{NMx^{b-r}L(x)}-1\right| \leq \left|\int_x^{\delta}\frac{(1+\in(t))Ct^{-(r-b)-1}L(t)dt}{Mx^{b-r}L(x)}-1\right|+\varepsilon/8$$

$$\leq\left|C\int_x^{\delta}\frac{t^{b-r-1}L(t)dt}{Mx^{b-r}L(x)}-1\right|+\varepsilon/8\frac{\left|\int_x^{\infty}t^{b-r-1}L(t)dt\right|}{Mx^{b-r}L(x)}+\varepsilon/8 \quad \left(\because \left|C\varepsilon(x)\right|\leq\varepsilon/8\right)$$

$$\leq\left|\frac{C\int_1^{\delta/x}u^{b-r-1}L(ux)du}{ML(x)}-1\right|+\varepsilon/8\frac{\left|\int_1^{\delta/x}u^{b-r-1}L(ux)du\right|}{ML(x)}+\varepsilon/8$$

$$\leq\left|C\frac{\int_1^{\infty}u^{b-r-1}L(ux)du}{ML(x)}-1\right|+C\frac{\left|\int_\delta^{\infty}t^{b-r-1}L(t)dt\right|}{Mx^{b-r}L(x)}+\varepsilon/8\frac{\left|\int_1^{\infty}u^{b-r-1}L(ux)du\right|}{ML(x)}+\varepsilon/8<\varepsilon$$

$$\therefore\left|\frac{\Gamma(r+1)(T_{r+1}f)(x)}{NMx^{b-r}L(x)}-1\right|<\varepsilon. \qquad (3.13)$$

We made use of the relation

$$\int_1^{\infty}u^{b-r-1}L(ux)du \sim L(x)\int_1^{\infty}u^{b-r-1}du,\ x\to 0^+$$

$$\therefore\left|\frac{\Gamma(r+1)(T_{r+1}f)(x)}{NMx^{b-r}L(x)}-1\right|=0 \qquad \text{(By 3.13)}$$

$$\therefore \frac{\Gamma(r+1)(T_{r+1}f)(x)}{NMx^{b-r}L(x)} = 1$$

$$\therefore \Gamma(r+1)(T_{r+1}f)(x) = NMx^{b-r}L(x)$$

$$\therefore \Gamma(r+1)(T_{r+1}f)(x) \sim C\frac{(r+1)}{(r-b)}.x^{-(r-b)}L(x),\ x \to 0^{+}$$

Hence the proof.

CONCULUSION

By applying Stieltjes transforms and method given in Pilipovic [3] for quasi-asymptotic behaviour of distribution. We obtain "Initial value Abelian theorem for Distributional Modifeid Stieltjes Transformations".

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PEXIDER DIFFERENCE OPERATOR IN L^P SPACES

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ABSTRACT

Assume that (G, +) is a complete measurable group with an finite measure λ and E is a Banach space. For $f, g, h : G \to E$ define the Pexider difference operator by $P(f, g, h)(x, y) : = f(x+y) - g(x) - (h)y$, $x, y \in G$.

In this paper we present some results about such Pexider operator. We prove theorem about the Hyers-Ulam-Rassias stability of Pexider operator. Moreover, theorems concerning continuity and the existence of continuous inverse operator to Pexider operator are established.

INTRODUCTION

Let E be a Banach space and let $(G,+,\Sigma,\lambda)$ be a semigroup with zero with a complete note identicaly zero measure λ and such that $\lambda(G) < +\infty$.

For any functions $f,g,h : G \to E$ we define Cauchy and Pexider operators respectively as follows

$$CF(x,y) : f(x+y) - f(x) - f(y),$$

$$P(f,g,h)(x,y) : \ = f(x+y) - g(x) - h(y).$$

Now we prove

Lemma 1.1 For every $f,g,h : G \to E$ we have

$$Cf(x,y) = P(f,g,h)(x,y) - P(f,g,h)(0,y) - P(f,g,h)(x,0) - h(0) - g(0), \qquad (1)$$

$$Cg(x,y) = P(f,g,h)(x,y) - P(f,g,h)(x+y,0) + P(f,g,h)(y,0) - P(f,g,h)(0,y) - g(0), \qquad (2)$$

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$$Ch(x,y)=P(f,g,h)(x,y)-P(f,g,h)(0,x+y)+P(f,g,h)(0,x)-P(f,g,h)(x,0)-h(0). \tag{3}$$

Proof. We get for $x,y\in G$,

$$\begin{aligned} Cf(\cdot\cdot,y)&=f(x+y)-f(x)-f(y)=f(x+y)-g(x)-h(y)+g(x)+h(0)\\ &\quad-f(x)+h(y)+g(0)-f(y)-g(0)-h(0)\\ &=P(f,g,h)(x,y)-P(f,g,h)(x,0)-P(f,g,h)(0,y)-g(0)-h(0). \end{aligned}$$

Analogously, we obtain

$$\begin{aligned} Cg(x,y)&=g(x+y)-g(x)-g(y)=g(x+y)+h(0)-f(x+y)\\ &\quad+f(x+y)-g(x)-h(y)+h(y)+g(0)-f(y)+f(y)-g(y)-h(0)\\ &=P(f,g,h)(x,y)-P(f,g,h)(x+y,0)+P(f,g,h)(y,0)-P(f,g,h)(0,y)-g(0). \end{aligned}$$

The proof of (3) is quite similar.

Now we establish some denotation and general hypothesis. We define

${}_yA=\{t\in G;y+t\in A\}$, $A_y=\{t\in G;t+y\in A\}$ for $A\in\Sigma$.

We assume :

(p) $A_y\in\Sigma$ and $\lambda(A_y)=\lambda(A)$ for all $A\in\Sigma$ and all $A\in G$,

(l) ${}_yA\in\Sigma$ and $\lambda({}_yA)=\lambda(A)$ for all $A\in\Sigma$ and all $y\in G$.

The following denotations will be used further on.

For a complete measure space (X,Σ,ν) and $1\le p\le+\infty$ we denote (see [1], [2], [5]) $L^p_\nu(X,E):=\{f:X\to E;\ f \text{ is } \Sigma \text{ measurable and } |f|_p<+\infty\}$, where $|f|_p := \left(\int_X \|f(x)\|^p dv(x)\right)^{1/p}$ for $1 \le p < +\infty$, $|f|_\infty := \operatorname{esssup}_{t \in X} \|f(t)\|$.

Of course, function f and g are equal if they are equal almost everywhere with respect to the measure v.

If $f : G \to E$ and $y \in G$, then we put $f^y(x) := f(y + x), f_y(x) := f(x + y)$. For $f : G \times G \to E$, we denote $f^0(x) := f(0, x), f_0(x) := f(x, 0)$, $f^+(x, y) := f(x + y, 0)$, $f_+(x, y) := f(0, x + y)$.

We will apply the following

Lemma 1.2 ([6] Let (G, Σ, λ) be a complete measure space such that $0 < \lambda(G) < +\infty$ and let $f \in L^1_\lambda(G, E)$. If (p) is satisfied, then for $y \in G$ $f_y \in L^1_\lambda(G, E)$ and $\int_G f_y(t) d\lambda(t) = \int_G f(t) d\lambda(t)$. Moreover, if (l) is satisfied, then for $y \in G f^y \in L^1_\lambda(G, E)$ and $\int_G f^y(t) d\lambda(t) = \int_G f(t) d\lambda(t)$.

STABILITY OF THE PEXIDER EQUATION

In this section we are going to give the positive answer to the problem of Hyers-Ulam-Rassias stability (see also [3], [4]) of the Pexider equation :

$$f(x + y) = g(x) + h(y), \; x, y \in G.$$

The problem can be formulated as follows.

Let $f, g, h : G \to E$ be such that $P(f, g, h) \in L^p_{\lambda \times \lambda}(G \times G, E)$. Find Constant $M \ge 0$ and an additive mapping $A : G \to E$ such that the following conditions are satisfied

$$|f - A|_p \le M \|P(f, g, h)\|_p,$$

$$|g - A|_p \le M \|P(f, g, h)\|_p,$$

$$|h-A|_p \le M\|P(f,g,h)\|_p.$$

Note that $\|P(f,g,h)\|_p := \left[\int_{G\times G}\|P(f,g,h)(x,y)\|^p d\lambda(x)d\lambda(y)\right]^{1/p}$.

For the Cauchy equation this problem has been solved in [6] in the case $\lambda(G)<+\infty$ and in the case $\lambda(G)=+\infty$ in [7].

By μ we understand the completion of the product of the measure λ times itself.

The following theorem, which we will apply latter on, can be found in [6].

Theorem 2.1 Let G be a semigroup and let (G,Σ,λ) be a complete measure space satisfying at least one of the conditions (p), (l). We assume that the mapping $s:G\times G\ni(x,y)\to(x,x+y)\in G\times G$ is measurable. Let E be a Banach space and let $f:G\to E$ be such that $Cf\in L^p_\mu(G\times G,E)$ for a certain $l\le p\le+\infty$. If $0<\lambda(G)<+\infty$, then there exists function $A:G\to E$ such that

(i) $CA=0$

(ii) $f-A\in L^p_\lambda(G,E)$,

(iii) $|f-A|_p \le \lambda(G)^{-1/p}\|Cf\|_p$, .

(iv) for every $A^*:G\to E$ satisfying the conditions $CA^*=0$, and $f-A^*\in L^p_\lambda(G,E)$, we have $A=A^*$.

The next result is the following

Theorem 2.2 Let the assumptions of Theorem 2.1 be satisfied. Moreover, Let $f,g,h:G\to E$ be such that $P(f,g,h),P(f,g,h)^+,P(f,g,h)_+\in L^p_\mu(G\times G,E)$,

$P(f,g,h)^0, P(f,g,h)_0 \in L_\lambda^p(G,E)$ for a fixed $1 \le p \le +\infty$. If $0 < \lambda(G) < +\infty$, then there exists $A : G \to E$ such that

(v) $CA = 0$

(vi) $f - A, g - A, h - A \in L_\lambda^p(G,E)$,

(vii) a) $|f - A|_p \le \lambda(G)^{-1/p} \|Cf\|_p$,

b) $|g - A|_p \le \lambda(G)^{-1/p} \|Cg\|_p$,

c) $|h - A|_p \le \lambda(G)^{-1/p} \|Ch\|_p$,

(viii) a) if $B : G \to E$, $CB = 0, f - B \in L_\lambda^p(G,E)$, then $B = A$,

b) if $D : G \to E$, $CD = 0$, $g - D \in L_\lambda^p(G,E)$, then $D = A$,

c) if $H : G \to E$, $CH = 0$, $h - H \in L_\lambda^p(G,E)$, then $H = A$.

Proof. At first we quess, in view on Lemma 1.1, that $Cf, Cg, Ch \in L_\mu^p(G \times G, E)$. Therefore, making use of Theorem 2.1, we obtain that there exist A, A_1, $A_2 : G \to E$ such that

$$CA = CA_1 = CA_2 = 0, \tag{4}$$

$$f - A, g - A_1, h - A_2 \in L_\lambda^p(G,E),$$

$$|f - A|_p \le \lambda(G)^{1/p} \|Cf\|_p,$$

$$|g - A_1|_p \le \lambda(G)^{-1/p} \|Cg\|_p, \tag{5}$$

$$|h - A_2|_p \le \lambda(G)^{-1/p} \|Ch\|_p.$$

We shall prove that $A = A_1 = A_2$ (with respect to the measure λ).

Note that we have

$$g(x)-A(x)=-P(f,g,h)(x,0)-h(0)+f(x)-A(x),$$

$$h(x)-A(x)=-P(f,g,h)(x,0)-g(0)+f(x)-A(x),$$

Whence, using our assumptions, we get

$$g-A,h-A\in L^p_\lambda(G,E). \tag{6}$$

From Theorem 2.1 we have

if $B:G\to E$ satisfies $CB=0$, $g-B\in L^p_\lambda(G,E)$, then $B=A_1$,

if $D:G\to E$ satisfies $CD=0$, $h-D\in L^p_\lambda(G,E)$, then $D=A_2$,

Hence, taking $B=A$ and $D=A$, in view of (4) and (6) we get $A=A_1=A_2$.

The conditions (vii) and (viii) we obtain now directly from (5) and (7) (substitiuting $A=A_1=A_2$). This completes the proof.

Corollay 2.1 Under The assumption of Theorem 2.2 istead of inequalities (vii) we have

(ix) a) $$|f-A|_p\le\lambda(G)^{-1/p}\|P(f,g,h)\|_p+\left|P(f,g,h)^0\right|_p$$

$$+\left|P(f,g,h)_0\right|_p+\lambda(G)^{1/p}(\|g(0)\|+\|h(0)\|),$$

b) $$|g-A|_p\le\lambda(G)^{-1/p}\|P(f,g,h)\|_p+\left|P(f,g,h)^+\right|_p+$$

$$\left|P(f,g,h)^0\right|_p+\left|P(f,g,h)_0\right|_p+\lambda(G)^{1/p}\|g(0)\|,$$

c) $$|h-A|_p\le\lambda(G)^{-1/p}\|P(f,g,h)\|_p+\left|P(f,g,h)_+\right|_p+$$

$$\left|P(f,g,h)^0\right|_p + \left|P(f,g,h)_0\right|_p + \lambda(G)^{1/p}\|h(0)\|.$$

Proof. We will prove the condition a). The inequalities b) and c) one can verify analogously. In virtue of Lemma 1.1 we have $Cf(x,y) = P(f,g,h)(x,y) - P(f,g,h)^0(x,y) - P(f,g,h)_0(x,y) - g(0) - h(0)$. Hence we get

$$\|Cf\|_p \le \|P(f,g,h)\|_p + \left\|P(f,g,h)^0\right\|_p + \|P(f,g,h)_0\|_p + \|g(0)\|_p + \|h(0)\|_p \le \|P(f,g,h)\|_p +$$

$$\lambda(G)^{1/p}\left|P(f,g,h)^0\right|_p + \lambda(G)^{1/p}\left|P(f,g,h)_0\right|_p + \lambda(G)^{2/p}\left[\|g(0)\| + \|h(0)\|\right].$$

Consequely, applying the inequality (vii) a), we obtain the condtion (ix) a), which completes the proof.

PEXIDER LINEAR OPERATOR

Recall the following theorem contained in [6].

Theorem 3.1 Let G be a non-empty set with a binary operation +. Let (G, Σ, λ) be a measure space satifying one of the conditions (p) or (l). If $0 < \lambda(G) < +\infty$, $f \in L_\lambda^p$ for a certain $1 \le p \le +\infty$ and the mapping $G^2 \ni (x,y) \to f(x+y)$ is measurable in the product space, than

(x) $Cf \in L_\mu^p(G \times G, E)$,

(xi) $|f|_p \le \lambda(G)^{-1/p}\|Cf\|_p$,

(xii) $\|Cf\|_p \le 3\lambda(G)^{1/p}|f|_p$.

Now we prove

Theorem 3.2 Let the assumptions of Theorem 3.1 be satisfied. If $f, g, h \in L_\lambda^p$ for a certain $1 \le p \le +\infty$, $0 \in G$ and the mapping $G^2 \ni (x,y) \to f(x+y)$,

$G^2 \ni (x,y) \to g(x+y)$, $G^2 \ni (x,y) \to h(x+y)$ are measurable in the product space, then

(xiii) Operator : $(f,g,h) \to P(f,g,h)$ is linear,

(xiv) $P(f,g,h)L^p_\mu(G \times G, E)$,

(xv) If $h(0) = g(0) = 0$, then

$$|f|_p + |g|_p + |h|_p \le 3\lambda(G)^{-1/p}\left(\|P(f,g,h)\|_p + \left\|P(f,g,h)^+\right\|_p + \left\|P(f,g,h)_+\right\|_p\right) +$$

$$3\left(\left|P(f,g,h)^0\right|_p + \left|P(f,g,h)_0\right|_p\right),$$

(xvi) If $h(0) = g(0) = 0$, then $\lambda(G)^{-1/p}\left(\|P(f,g,h)\|_p + \left\|P(f,g,h)^+\right\|_p + \left\|P(f,g,h)_+\right\|_p\right)$

$$+\left|P(f,g,h)^0\right|_p + \left|P(f,g,h)_0\right|_p \le 5\left(|f|_p + |g|_p + |h|_p\right).$$

Proof. If $f,g,h,k,l,m \in L^p_\lambda$, and $s \in K$ (set of real or complex number), then we get easily

$$P(f+k, g+l, h+m) = P(f,g,h) + P(k,l,m),$$

$$P(sf, sg, sh) = sP(f,g,h)$$

so P is linear operator.

Let $l_r : G \times G \to E$, $r = 1,2,3$ be defined by : $l_1(x,y) := f(x+y)$, $l_2(x,y) : g(x)$, $l_3(x,y) : h(y)$. Now we have by the Fubini-Tonelli theorem

$$\|l_2\|_p^p = \int_{G \times G} \|l_2(x,y)\|^p d\mu(x,y) = \int_{G \times G} \|g(x)\|^p d\mu(x,y) =$$

$$\int_G d\lambda(y) \int_G \|g(x)\|^p d\lambda(x) = \lambda(G)|g|_p^p < +\infty$$

and hence $l_2 \in L^p_\mu$. By the same arguments we get $l_3 \in L^p_\mu$ and $\|l_3\|_p^p = \lambda(G)|h|_p^p$.

Similarly in view of lemma 1.2 we obtain

$$|f|_p^p = \int_G \|f(x)\|^p d\lambda(x) = \int_G \|f(x+y)\|^p d\lambda(x) = \lambda(G)^{-1}\int_G d\lambda(y)\int_G \|l_1(x,y)\|^p d\lambda(x) =$$

$$\lambda(G)^{-1} \int_{G\times G} \|l_1(x,y)\|^p d\lambda(x,y) = \lambda(G)^{-1}\|l_1\|_p^p,$$

whence $|f|_p = \lambda(G)^{-1/p}\|l_1\|_p$. Therefore $P(f,g,h) = l_1 - l_2 - l_3 \in L^p_\mu$ and (xiv) has been proved.

Consequently we have also proved that

$$\|P(f,g,h)\|_p \le \|l_1\|_p + \|l_2\|_p + \|l_3\|_p = \lambda(G)^{1/p}\left(|f|_p + |g|_p + |h|_p\right). \tag{8}$$

Now we shall prove (xv). From Theorem 3.1 we have the inequalities

$$|f|_p \le \lambda(G)^{-1/p}\|Cf\|_p, |g|_p \le \lambda(G)^{-1/p}\|Cg\|_p, |h|_p \le \lambda(G)^{-1/p}\|Ch\|_p.$$

Therefore, by Lemma 1.1, we obtain successively

$$|f|_p + |g|_p + |h|_p \le \lambda(G)^{-1/p}\left(\|Cf\|_p + \|Cg\|_p + \|Ch\|_p\right) \le \lambda(G)^{-1/p}\Big(\left\|P(f,g,h) - P(f,g,h)^0 - P(f,g,h)_0\right\|_p$$

$$+\left\|P(f,g,h) - P(f,g,h)^+ - P(f,g,h)^0 + P(f,g,h)_0\right\|_p + \Big\|P(f,g,h) - P(f,g,h)_+ -$$

$$-P(f,g,h)_0 + P(f,g,h)^0\Big\|_p\Big) \le 3\lambda(G)^{-1/p}\Big(\|P(f,g,h)\|_p + \left\|P(f,g,h)^+\right\|_p + \left\|P(f,g,h)_+\right\|_p\Big) +$$

$$+3\Big(\left|P(f,g,h)^0\right|_p + \left|P(f,g,h)_0\right|_p\Big).$$

Similarly one can easily check that

$$P(f,g,h)^0\Big|_p \le |f|_p + |h|_p, \ \left|P(f,g,h)_0\right|_p \le |f|_p + |g|_p, \ \left\|P(f,g,h)^+\right\|_p \le \lambda(G)^{1/p}$$

$$\left(|f|_p+|g|_p\right),\ \left\|P(f,g,h)_+\right\|_p \le \lambda(G)^{1/p}\left(|f|_p+|h|_p\right).$$

Adding the above inequalities and the inequality (8), we get (xvi), which proves the theorem.

Let's introduce some definitions.

$$DP:=\left\{(f,g,h)\in L^p_\lambda\times L^p_\lambda\times L^p_\lambda;\, P(f,g,h)^+\in L^p_\mu, P(f,g,h)_+\in L^p_\mu, P(f,g,h)\in L^p_\mu,\right.$$

$$\left. h(0)=0, g(0)=0\right\}$$

In DP we define the norm : $\|(f,g,h)\|:=|f|_p+|g|_p+|h|_p,(f,g,h)\in DP,$

and in L^p_μ the following one : $\|F\|:=\lambda(G)^{-1/p}\left(\|F\|_p+\|F^+\|_p F_+\right)+$

$$\left|F^0\right|_p+\left|F_0\right|_p, F\in L^p_\mu.$$

Remark 3.1 The set DP is a linear supspace of $L^p_\lambda(G,E)\times L^p_\lambda(G,E)\times L^p_\lambda(G,E)$.

Remark 3.2 The following equality $DP=\{(f,g,h)\in L^p_\lambda\times L^p_\lambda\times L^p_\lambda;(x,y)\to f(x+y),(x,y)\to g(x+y),(x,y)\to h(x+y)$ are measurable in the product space$\}$ is the immadiate consequence of the properties of measurable functions and our Theorem 3.2.

Now we are in the position to present.

Theorem 3.3 Let the assumptions of Theorem 3.2 be satisfied and $g(0)=h(0)=0$. Then the linear operator $P:DP\ni(f,g,h)\to P(f,g,h)\in L^p_\lambda(G\times G,E)$ is continuous and continuously invertible. Moreover, the inverse operator (defined for all $m\in P(DP)$ has the form :

$$P^{-1}m(.)=\left(-\lambda(G)^{-1}\int_G\left[m(x,.)-m_0(x)-m^0(.)\right]d\lambda(x),-\lambda(G)^{-1}\int_G\left[m(x,\cdot)-m^0(\cdot)\right.\right.$$

$$-m^0(.)+m_0(.)\big]d\lambda(x),-\lambda(G)^{-1}\int_G\big[m(x,.)-m_+(x,.)-m_0(x)+m^0(x)\big]d\lambda(x)\big)$$ if (p) occurs and

$$P^{-1}m(.)=\Big(-\lambda(G)^{-1}\int_G\big[m(.,y)-m_0(.)-m^0(y)\big]d\lambda(y),-\lambda(G)^{-1}\int_G\big[m(.,y)-m^+$$

$$(.,y)-m^0(y)+m_0(y)\big]d\lambda(y),-\lambda(G)^{-1}\int_G\big[m(.,y)-m_+(.,y)-m_0(.)+m^0(.)\big]d\lambda(y)\Big)$$ if (l) occurs.

Proof. The continuity of P follows from Theorem 3.2 point (xvi) and Remarks 3.2. The existence of inverse operator P^{-1} are the consequence of Remark 3.2, Theorem 3.2 point (xv) and Theorem 3, *p*. 43 from [8].

Now we establish the form of p^{-1}. Let us assume the condition (p) (in the case (1) the proof runs quite similarly). Note that $Cf, Cg, Ch \in L^1_\lambda(G,E)$ and in view of **Lemma 1.2** we have $f(y)=-\lambda(G)^{-1}\int_G Cf(x,y)d\lambda(x)$, $g(y)=-\lambda(G)^{-1}\int_G Cg(x,y)d\lambda(x)$, $h(y)=-\lambda(G)^{-1}\int_G Ch(x,y)d\lambda(x)$.

Making use of Lemma1.1 we obtain equalities

$$f(y)=-\lambda(G)^{-1}\int_G\big(P(f,g,h)(x,y)-P(f,g,h)(x,0)-P(f,g,h)(0,y)\big)d\lambda(x),$$

$$g(y)=-\lambda(G)^{-1}\int_G\big(P(f,g,h)(x,y)-P(f,g,h)(x+y,0)$$
$$+P(f,g,h)(y,0)-P(f,g,h)(0,y)\big)d\lambda(x),$$

$$h(y)=-\lambda(G)^{-1}\int_G\big(P(f,g,h)(x,y)-P(f,g,h)(0,x+y)+$$
$$P(f,g,h)(0,x)-P(f,g,h)(x,0)\big)d\lambda(x).$$

Hence we obtain the form of P^{-1}. This completes the proof.

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